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Linear Approximations and Differentials

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Some Useful Background:

(1) Given: $y = f(x) : \mathbb{R} \to \mathbb{R}$

its Derivative:

$$y' = f'(x) = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}$$

This is the most "Basic" Definition of Derivative

(2) Linear Approximation:

Taylor Series:

$$f(x) = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a)$$

We must smartly select "a".

How to select "a"?

This is Beyond the scope of this Note

Keyword: "R.o.C"

Linear Evolution (around "a")

$$f(x) \simeq f(a) + (x-a) f'(a)$$

$f(a)$: $i = 0$; $f'(a)$: $i = 1$

Ex. 2 $f(x) = \sqrt{3x+1}$

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(i) find $f'(x)$ ✓

(ii) find $f'(x)$ using definition ✓

(iii) find linear Approximation of $f(x)$ with $a=1$ ✓

(iv) use linear Approximation to estimate $\sqrt{5}$ ✓

(v) use Differential to estimate $\sqrt{9.07}$ ✓

(i) $f'(x) = \left(\sqrt{3x+1}\right)' = \left((3x+1)^{1/2}\right)' = \frac{1}{2}(3x+1)^{-1/2}(3)$

$$= \frac{3}{2}\frac{1}{\sqrt{3x+1}}$$

$$\boxed{f'(x) = \frac{3}{2\sqrt{3x+1}}}$$

(ii) $$f'(x) = \lim_{h\to 0} \frac{(f(x+h) - f(x))}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{3(x+h)+1} - \sqrt{3x+1}\right)}{h} = \frac{0}{0} \quad \text{I.F.}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{3(x+h)+1} - \sqrt{3x+1}\right)}{h} \left(\frac{\sqrt{3(x+h)+1} + \sqrt{3x+1}}{\sqrt{3(x+h)+1} + \sqrt{3x+1}}\right)$$

Conjugate

$$= \lim_{h\to 0} \frac{(3(x+h)+1) - (3x+1)}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} = \text{---} \longrightarrow \text{Next Page!}$$

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Continue of (ii)

$$f'(x) = \lim_{h\to 0} \frac{(3x+3h+1-3x-1)}{h\left(\sqrt{3(x+h)+1}+\sqrt{3x+1}\right)}$$

$$= \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3(x+h)+1}+\sqrt{3x+1}\right)}$$

$$= 3\lim_{h\to 0} \frac{1}{\left(\sqrt{3(x+h)+1}+\sqrt{3x+1}\right)} = \frac{3}{\left(\sqrt{3x+1}+\sqrt{3x+1}\right)}$$

$$= \frac{3}{2\sqrt{3x+1}}$$ ← Same Answer as "i" ☺

(iii) Linear approx: $f(x) \simeq f(a) + (x-a)f'(a)$

$$= f(1) + (x-1)f'(1)$$

* $f(1) = \sqrt{3+1} = \sqrt{4} = 2$
* $f'(1) = \frac{3}{2\sqrt{4}} = \frac{3}{2\cdot 2} = \frac{3}{4}$

$$= 2 + (x-1)\frac{3}{4}$$

$$= 2 + \frac{3}{4}x - \frac{3}{4}$$

$$= \frac{5}{4} + \frac{3}{4}x = \frac{(5+3x)}{4}$$

Linear Approximation

$$f(x) \simeq \frac{(5+3x)}{4}$$

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(iv) Estimate $\sqrt{5}$ using linear Approximation

$$f(x) = \sqrt{3x+1} = \sqrt{5}$$

$$3x+1 = 5$$

$$3x = 4$$

$$\boxed{x = \frac{4}{3}}$$

$$f(4/3) \simeq \frac{(5+3x)}{4} = \frac{[5+(3)(4/3)]}{4} = \frac{(5+4)}{4} = \frac{9}{4}$$

$$= 2.25$$

$$\boxed{f(4/3) = \sqrt{5} \simeq 2.25}$$

$\sqrt{5} \simeq 2.236\ldots$ (linear approx.)

overall the approx. error

$\boxed{\varepsilon} = 2.25 - 2.236\ldots$

$= \boxed{0.014}$ Not Bad!

(v) 1° estimate $\sqrt{9.07} \rightsquigarrow \sqrt{9} = 3 = \sqrt{3x+1} = f(x)$

$9 = 3x+1$

$8 = 3x \Rightarrow \boxed{x = 8/3} = 2.667\ldots$

2° $9.07 = 3x+1$

$\Rightarrow \boxed{x} = \frac{8.07}{3} = \boxed{2.69}$

3° $\boxed{\Delta x} = 2.69 - 8/3 \simeq \boxed{0.023}$

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3°

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x+1}}$$

approx.

$$\frac{\Delta y}{\Delta x} = \frac{3}{2\sqrt{3x+1}}$$

$\Delta x \simeq 0.023$, $x = 8/3$

$$\therefore \Delta y = \frac{(3)(0.023)}{(2)(3)} = \frac{0.023}{2} = 0.01167$$

4°

$$f(x+\Delta x) \simeq \Delta y + f(x) = (0.01167) + f(8/3)$$

$$= 3 + 0.01167$$

$$= 3.01167$$

Answer!

$$\therefore f(2.69) \simeq 3.01167$$

$$= \sqrt{9.07}$$

extraction

$$\sqrt{9.07} = 3.011644\ldots$$

exact

error

$$\varepsilon = 0.000026$$

Excellent !!!

WOW!

the end

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EX2 Estimate $\sqrt{25.05}$

1° Say: $f(x) = \sqrt{x}$

2° $f'(x) = (x^{1/2})' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

3° $\sqrt{25.05} \rightsquigarrow \sqrt{25} = f(25)$ $\quad x = 25$ $\quad \therefore \Delta x = 25.05 - 25 = 0.05$

4° $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ estimate $\Rightarrow$ $\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}$

$\Delta y = \frac{\Delta x}{2\sqrt{x}} = \frac{(0.05)}{(2)\sqrt{25}}$

5° $f(x + \Delta x) \simeq \Delta y + f(x)$ (25, 0.05, 0.005, 25)

$= \frac{(0.05)}{(2)(5)} = \frac{0.05}{10} = 0.005$

$\therefore f(25.05) = \sqrt{25.05} \simeq 0.005 + 5 = 5.005$

$\therefore \sqrt{25.05} \simeq 5.005$

$\sqrt{25.05} = 5.004997502$

error: $\varepsilon = 0.0000025$

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EX#3 Estimate $\sqrt{99.8}$

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1° $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}}$$

2° $\sqrt{100} = f(100) \rightsquigarrow \sqrt{99.8} = f(99.8)$

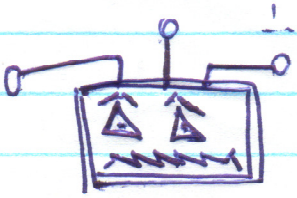
$$\boxed{\Delta x} = 100 + 99.8 = \boxed{0.2}$$

3° $$\boxed{\Delta y} = \frac{\Delta x}{2\sqrt{x}} = \frac{-(0.2)}{(2)\sqrt{100}} = \frac{-(0.2)}{(2)(10)} = \frac{-0.2}{20} = \boxed{0.01}$$

4° $f(x + \Delta x) \simeq \Delta y + f(x)$

(x: 100, Δx: −0.2, Δy: −0.01)

$$\therefore f(99.8) = \sqrt{99.8} \simeq (-0.01) + 10 = 9.99$$



$$\boxed{\sqrt{99.8} \simeq 9.99}$$

$\varepsilon \simeq 0.000005$

↑ Not Bad!

Thank.