On "Fourier Series"

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July. 11.2013

Consider the signal $x(t) = \sum_{k=-\infty}^{\infty} x_1(t-4k)$, $x_1(t) = \begin{cases} -1, & -2 < t < 0 \\ 1, & 0 < t < 2 \\ 0, & otherwise \end{cases}$

(a) Evaluate Fourier series coefficients a_k of the signal $x(t)$

(b) Sketch the frequency domain representation of the Fourier series

(c) Evaluate : $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

(d) Evaluate Fourier series coefficients b_k of the signal $y(t)$

(e) Sketch the frequency domain representation of the Fourier series b_k

(f) Evaluate the quantity $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$ (Hint: evaluate $y(0)$ using Fourier series)

(g) Evaluate the quantity $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$ (Hint: and apply Parseval's relation to $y(t)$)

(h) Evaluate and sketch the CT Fourier Transform of the signal $y(t)$

→ The objective of this "teaching Note" is to solve the above Problem.

→ However, various practical material will also be shown as we progressively solve these questions.

→ Also, some side but relevant material will be discussed.

OK, let's get to Work!

Important 1

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Before We Start the Problems Here are Some Useful Stuff!

1°

Horizontal Shift ($t \mp t_0$) $\xleftarrow{\text{left}}$ $\xrightarrow[\text{right}]{}$ only affects a_k ($k \neq 0$)

Vertical Shift ($x(t) \pm A$) $\uparrow$ top $\downarrow$ down only affects a_0 ($k=0$)

2°

If:
- → $x(t)$ is periodic
- → $x(t)$ is real
- → $x(t)$ is odd

Then:
- → "a_k" are "purely imaginary"
- → "a_k" are "odd"

$\boxed{Im\{a_k\} = -Im\{a_{-k}\}}$

3°

If:
- → $x(t)$ is periodic
- → $x(t)$ is real
- → $x(t)$ is even

Then:
- → "a_k" are "real"
- → "a_k" is "even"

$\boxed{a_k = a_{-k}}$

4°

(Even) x (Even) = Even
(odd) x (odd) = Even
(Even) x (odd) = odd

$$\int_{T_0} (\text{odd}) = 0$$

$$\int_{T_0} \text{even} = 2\int_{T_0/2} \text{even}$$

↑ it may become useful during analysis ... shortcuts are always handy!

Important 2

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5°

$$x(t) \overset{F.S.}{\Longleftrightarrow} a_K$$

$$y(t) = \int_{t=-\infty}^{t} x(t)\,dt \overset{F.S.}{\Longleftrightarrow} b_K = \frac{a_K}{jK\omega_0}$$ (only Valid for $K \neq 0$!!!)

$$b_0 = \frac{1}{T_0}\int_{T_0} y(t)\,dt = \frac{(\text{Area under } y(t) \text{ over } T_0)}{T_0}$$

(*)(*)(*) Very Important to Remember or understand

6°

What is "Parseval's Relation" for a "Periodic Signal"?

$$\boxed{\frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \sum_{K=-\infty}^{\infty} |a_K|^2}$$

Fourier Series

7°

$$x(t) : \text{periodic} \overset{F.S.}{\Longleftrightarrow} a_K$$

$$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t} \overset{F.T.}{\Longleftrightarrow} X(j\omega) = 2\pi \sum_{K=-\infty}^{\infty} a_K \,\delta(\omega - K\omega_0)$$

* Special --- Special --- Special !!!!!!!!!!!!!!!!!

Fourier Series

It's very useful to know this Somewhat "Weird" Reality !!

$\boxed{1}$

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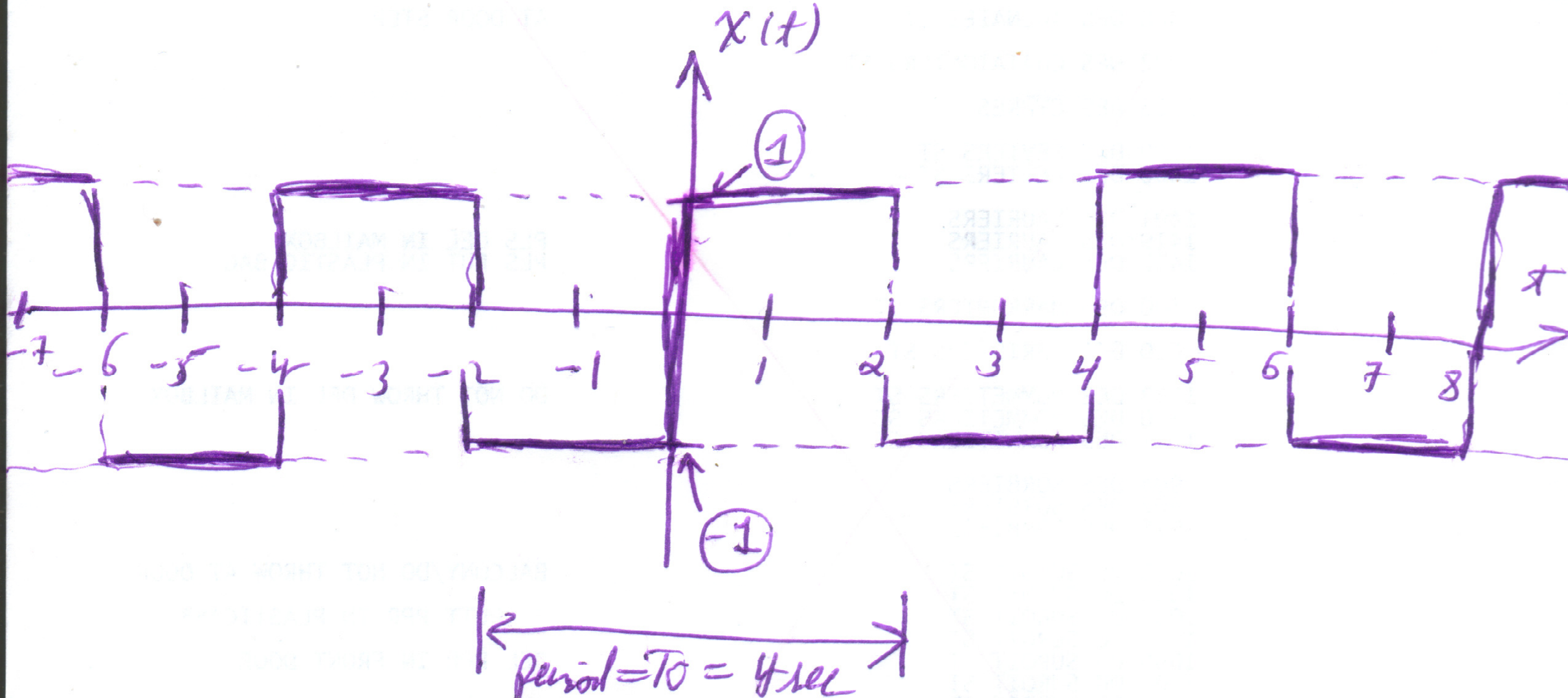
1° Sketch:

$$x(t) = \sum_{k=-\infty}^{\infty} x_0(t-4k)$$

Periodic signal with period $= T_0 = 4$

$$x_0(t) = \begin{cases} -1 & -2<t<0 \\ 1 & 0<t<2 \\ 0 & \text{otherwise} \end{cases}$$



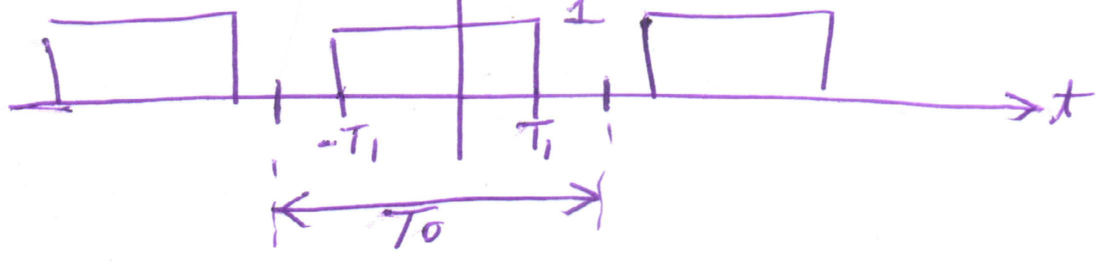
2° Find F.S. of x(t)

A

→ signal is periodic, ∴ F.S. does exist! ☺

→ x(t) looks very familiar!!

SquareWave signal



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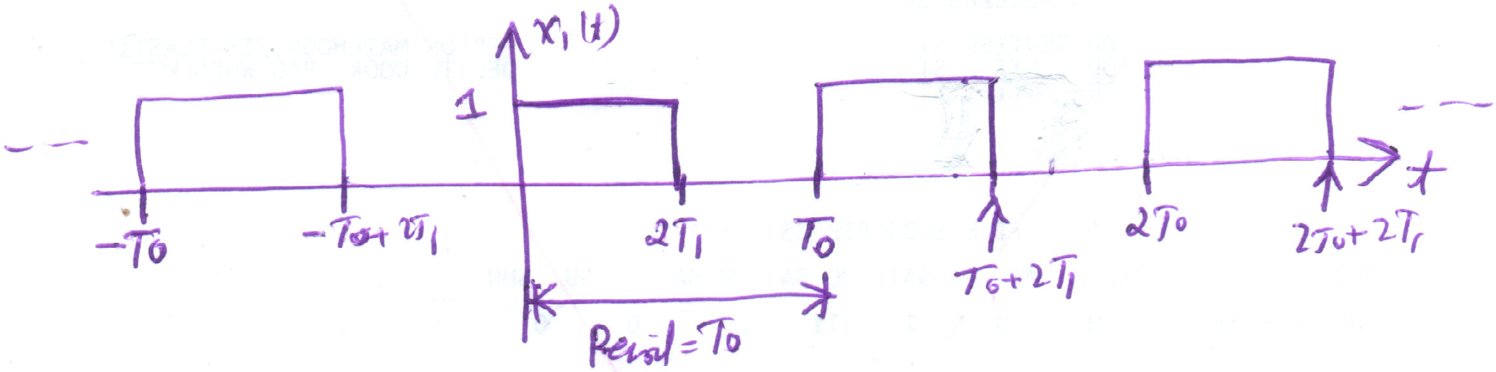
How to go from $x_s(t)$ to $x(t)$?

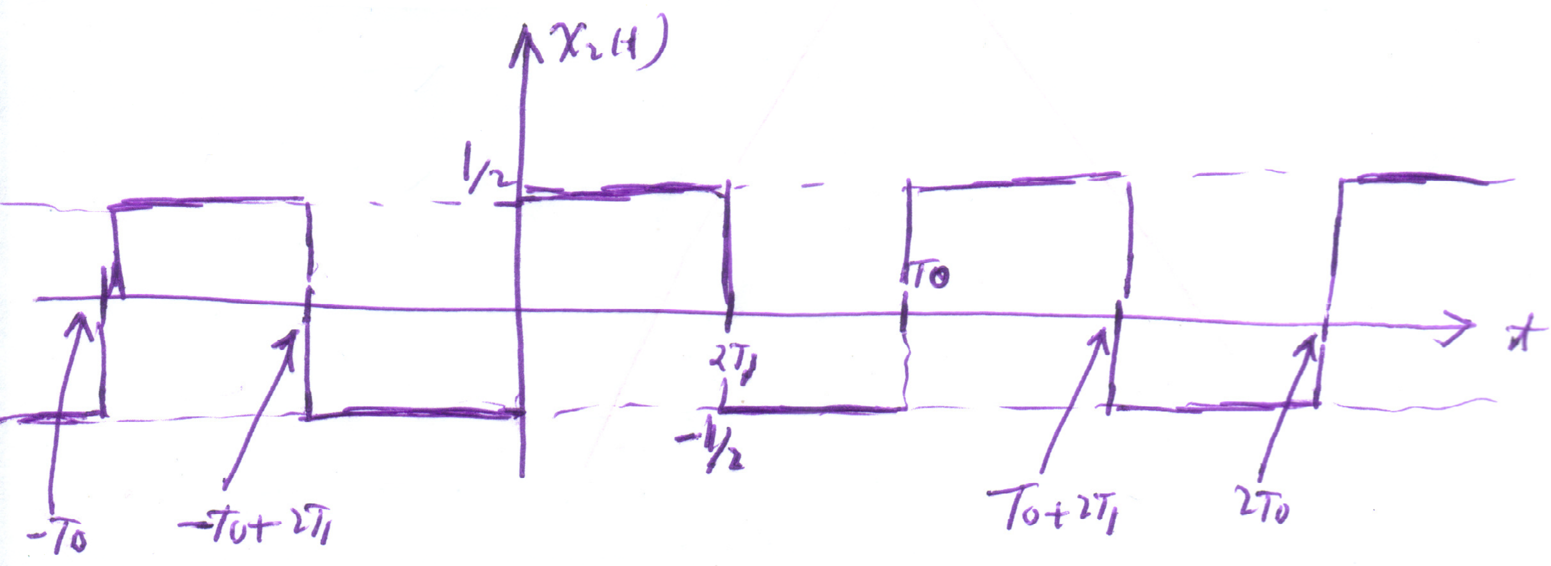
1° shift "$x_s(t)$" to the right by "T_1"

$$\rightarrow \boxed{x_1(t) = x_s(t - T_1)}$$



2° shift $x_1(t)$ down by "1/2"

$$\rightarrow \boxed{x_2(t)} = x_1(t) - 1/2$$
$$= \boxed{x_s(t - T_1) - 1/2}$$



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3° To Make "$x_2(t)$" similar to "$x(t)$" We need to make some very minor adjustments

→ i) $\boxed{T_1 = 1}$
→ ii) $\boxed{T_0 = 4}$
→ iii) Multiply the Amplitude by "2"

$$\boxed{x_2(t) = x_s(t-1) - 1/2}$$

$$\therefore \boxed{x(t)} = 2x_2(t)$$

$$= 2\left[x_s(t-1) - 1/2\right]$$

$$= \boxed{2x_s(t-1) - 1}$$

Period $= 4$

So How Can We get "a_k"?

$\boxed{x_s(t) : \text{Square Wave}}$ ⟷ $\boxed{a_0 = (2T_1/T_0) \quad a_k = \frac{\sin(2\pi k T_1/T_0)}{\pi k}}$

set $T_1 = 1$, $T_0 = 4$ $\therefore$ $a_0 = \frac{2 \cdot 1}{4} = \frac{1}{2}$

$$a_k = \frac{\sin\left(\frac{2\pi k \cdot 1}{4}\right)}{\pi k}$$

$$= \frac{\sin(\pi k/2)}{\pi k}$$

No:

$\boxed{x_s(t) : \text{Square Wave} \quad T_1 = 1 \quad T_0 = 4}$ $\overset{\text{f.s.}}{\Longleftrightarrow}$ $\boxed{a_0 = 1/2 \quad a_k = \frac{\sin(\pi k/2)}{\pi k}}$

4

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$$\boxed{x_s(t-1)} \Longleftrightarrow a_k e^{-jk\omega_0 t_0}$$

$$= a_k e^{-jk\frac{\pi}{2}(1)}$$

$$\boxed{\omega_0} = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

$$= \boxed{a_k e^{-jk\frac{\pi}{2}}}$$

change on "t" again
Vary $k \neq 0$

$$\boxed{e^{-jk\pi/2}} = \left(e^{-j\pi/2}\right)^k = \left(e^{-j90^\circ}\right)^k = \boxed{(-j)^k}$$

$$\boxed{x_s(t-1)} \overset{F.S.}{\Longleftrightarrow} \boxed{\begin{aligned} a_0 &= 1/2 \\ a_k &= \underbrace{e^{-jk\pi/2}}_{\boxed{(-j)^k}} \frac{\sin(\pi k/2)}{\pi k} \end{aligned}}$$

$$\boxed{2x_s(t-1)} \overset{F.S.}{\Longleftrightarrow} \boxed{\begin{aligned} a_0 &= 1 \\ a_k &= 2(-j)^k \frac{\sin(\pi k/2)}{\pi k} \end{aligned}}$$

Final Answer!!

$$\boxed{x(t) = 2x_s(t-1) - 1} \overset{F.S.}{\Longleftrightarrow} \boxed{\begin{aligned} a_0 &= 0 \\ a_k &= 2(-j)^k \frac{\sin(\pi k/2)}{\pi k} \end{aligned}}$$

Shift in the x(t)
Vertical
∴ only Vary $k=0$

Method #1

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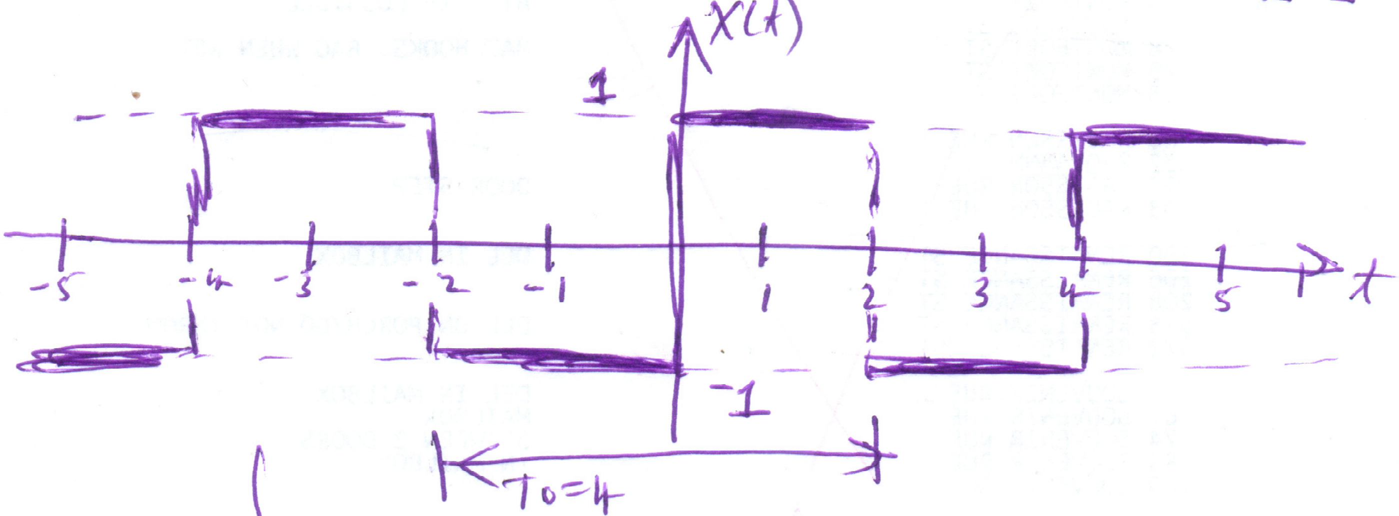
Could we have obtained the F.S. of $x(t)$ on Page 1 without the knowledge of "a_k" for a Square Wave?

→ **Yes** → We will do it **Now**



1) This signal is **Periodic**

2) this signal is Real and "**odd**" $\boxed{x(t) = -x(-t)}$

∴ "a_k" of this function are **purely imaginary** and **odd**

$\boxed{6}$

$\boxed{x(t) \text{ is real}} \Longleftrightarrow \boxed{a_k = a^*_{-k}}$

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$$\boxed{a_0} = \frac{1}{T_0} \int_{T_0} x(t)\, dt \ldots$$

$$= \frac{1}{4} \int_{t=-2}^{2} x(t)\, dt$$

$$= \frac{1}{4} \underbrace{\left[(-2) + (2)\right]}_{\text{area under the curve!}} = \boxed{0}$$

$\boxed{e^{j\theta} = \cos(\theta) + j \sin(\theta)}$

$$a_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$\boxed{\cos(\theta) = \cos(-\theta)}$

$$= \frac{1}{T_0} \int_{T_0} x(t) \left(\cos(-k\omega_0 t) + j \sin(-k\omega_0 t)\right) dt$$

$\boxed{\sin(-\theta) = -\sin(\theta)}$

$$= \frac{1}{T_0} \int_{T_0} x(t) \left(\cancel{\cos(k\omega_0 t)} - j \sin(k\omega_0 t)\right) dt$$

As mentioned on Page 5 if $x(t)$ is real & odd then "a_k" are purely imaginary !!!

$\boxed{7}$

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So: $a_k = \frac{-j}{T_0}\int_{T_0} \underbrace{x(t)}_{\text{odd}}\ \underbrace{\sin(k\overset{\pi/2}{\omega_0} t)}_{\text{odd}}\, dt$

1° (odd) x (odd) = (even)

2° $\boxed{\int_{T_0} \text{even} = 2\int^{T_0/2} \text{even}}$

= odd

so as expected "a_k" is odd!

☺

extra stuff

$\int_{T_0} \text{odd} = 0$

Memorize!

or simply understand

$= \frac{-2j}{T_0}\int_0^{T_0/2 = 2} \sin(k\frac{\pi}{2}t)\,dt$

Let: $u = \frac{k\pi}{2}t$

$du = \frac{k\pi}{2}dt$

$(\sin(x))' = \cos(x)$

$(\cos(x))' = -\sin(x)$

$\int \sin(x)dx = -\cos(x)$

$= \frac{-2j}{T_0}\int \sin(u)\ \frac{2}{k\pi}\,du$

$= \frac{-\cancel{2}j}{\cancel{4}}\ \frac{\cancel{2}}{k\pi}\int \sin(u)\,du$

$= (-j/k\pi)\int \sin(u)\,du = (+j/k\pi)\left[\cos\left(\frac{k\pi}{2}t\right)\right]_0^2$

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$$a_k = \frac{j}{k\pi}\left[\cos\left(\frac{k\pi}{2}t\right)\right]_{t=0}^{t=2}$$

$$= \frac{j}{k\pi}\left[\cos\left(\frac{k\pi}{2}2\right) - \cos(0)\right]$$

$$= \frac{j}{k\pi}\left[\cos(k\pi) - 1\right]$$

$$= \frac{j}{k\pi}\left[(-1)^k - 1\right]$$

if $k = +ve$
$= 1, 2, 3, \ldots$

$\cos(\pi) = -1$
$\cos(2\pi) = 1$
$\cos(3\pi) = -1$

$\therefore \boxed{\cos(k\pi) = (-1)^k}$

if $k = -ve$
$= -1, -2, -3$

Well: $\cos(-k\pi) = \cos(k\pi)$

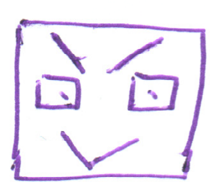
☺

Method #2

So

$$\boxed{\begin{aligned} a_0 &= 0 \\ a_k &= \frac{j}{k\pi}\left[(-1)^k - 1\right] \end{aligned}}$$

Is this the same as the result of Page $\boxed{4}$?



?

Yes it is!

Showing it may take some time! Will skip!

$\boxed{19}$

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sketch Frequency domain of $x(t)$ for $k=-3,-2,-1,0,1,2,3.$

$$x(t) \Longleftrightarrow a_k = \begin{cases} 0 \\ \dfrac{j}{k\pi}\left[(-1)^k - 1\right] \end{cases}$$

$k=-3$: $\boxed{a_{-3}} = \dfrac{j}{-3\pi}\left[(-1)^{-3}-1\right] = \dfrac{j}{-3\pi}\left[-1-1\right] = \dfrac{-2j}{-3\pi} = \boxed{\dfrac{2j}{3\pi}}$

$k=-2$: $\boxed{a_{-2}} = \dfrac{j}{-2\pi}\left[(-1)^{-2}-1\right] = \dfrac{j}{-2\pi}\left[1-1\right] = \boxed{0}$

Know this

$k=-1$: $\boxed{a_{-1}} = \dfrac{j}{-\pi}\left[(-1)^{-1}-1\right] = \dfrac{j}{-\pi}\left[-1-1\right] = \dfrac{-2j}{-\pi} = \boxed{\dfrac{2j}{\pi}}$

$k=0$: $\boxed{a_0 = 0}$

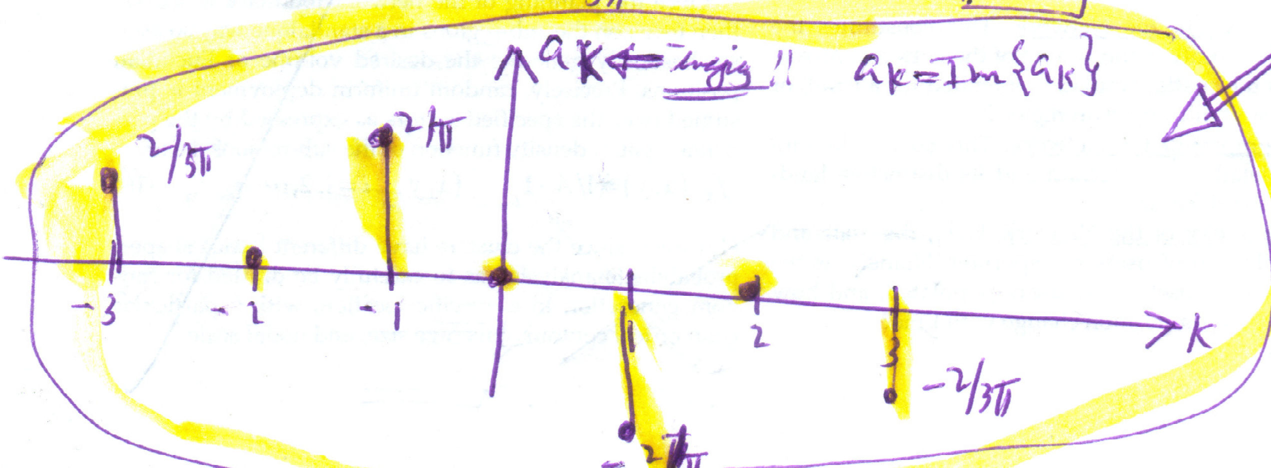
$k=1$: $\boxed{a_1} = \dfrac{j}{\pi}\left[(-1)^1-1\right] = \dfrac{j}{\pi}(-2) = \boxed{\dfrac{-2j}{\pi}}$

If $x(t)$ is real and odd $\Rightarrow$ "a_k" are imaginary and odd

$k=2$: $\boxed{a_2} = \dfrac{j}{2\pi}\left[(-1)^2-1\right] = \dfrac{j}{2\pi}\left[1-1\right] = \boxed{0}$

$k=3$: $\boxed{a_3} = \dfrac{j}{3\pi}\left[(-1)^3-1\right] = \dfrac{j}{3\pi}\left[-1-1\right] = \boxed{\dfrac{-2j}{3\pi}}$

As expected "a_k" is ODD



$\boxed{\boxed{P}}$

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for $x(t)$ of Page 1

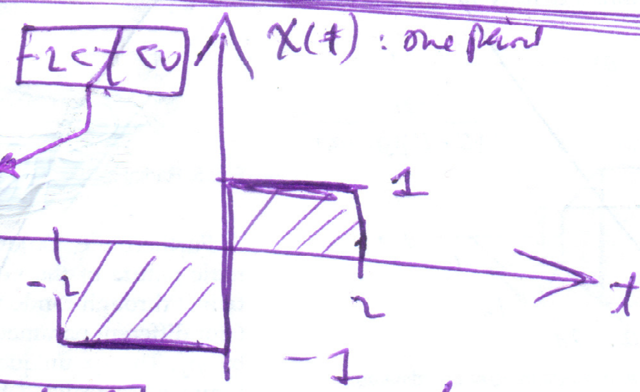
Show that:

$$y(t) = \int_{t=-\infty}^{t} x(t)\,dt$$

$$= \sum_{k=-\infty}^{\infty} y_0(t-4k)$$

$$y_0(t) = \begin{cases} |t| - 2 & |t| < 2 \\ 0 & \text{otherwise} \end{cases}$$

$x(t)$: one period



$\boxed{y(t)_{-ve}} = \int_{t=-2}^{t} (-1)\,dt = \left[-t\right]_{-2}^{t}$ $\boxed{-2 < t < 0}$

$= [t]_{t}^{-2} = \boxed{-2-t}$

$\boxed{y_{+ve}(t)} = \int_{t=-2}^{t} x(t)\,dt$

$\boxed{0 < t < 2}$

$= \int_{t=-2}^{0} (-1)\,dt + \int_{t=0}^{t} 1\,dt$

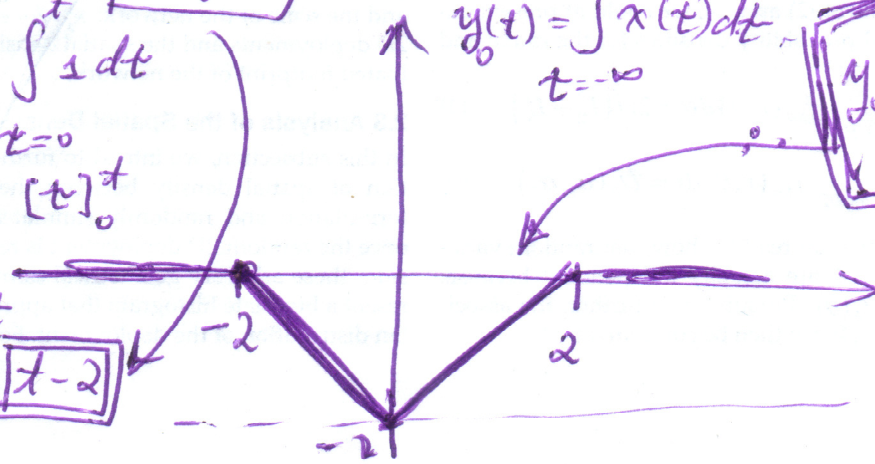
$= (-1)[t]_{-2}^{0} + [t]_{0}^{t}$

$= [t]_{0}^{-2} + t$

$= -2 + t = \boxed{\boxed{t-2}}$

$y_0(t) = \int_{t=-\infty}^{t} x(t)\,dt$

$\therefore$ $y_0(t) = \begin{cases} |t| - 2 & 0 < |t| < 2 \\ 0 & \text{otherwise} \end{cases}$



$\boxed{11}$

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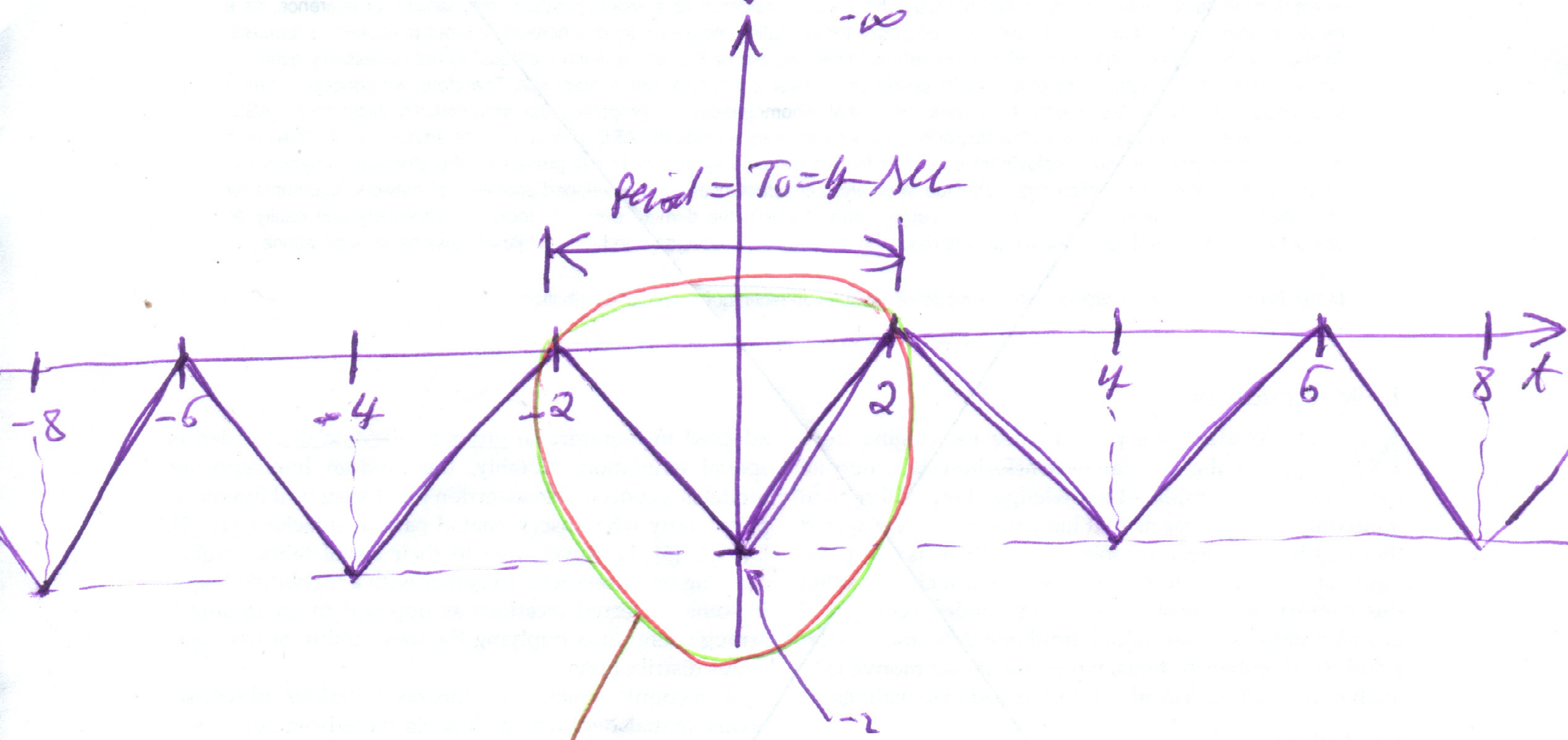
Since $x(t)$ is periodic with period "4" then $y(t) = \int_{-\infty}^{t} x(t)\,dt$ must also be periodic



$$= (|t| - 2) \cdot 1(|t| < 2)$$

$$\boxed{y_0(t) = |t| - 2 \quad 0 < |t| < 2}$$

(C)

so $\boxed{y(t)} = \underbrace{\sum_{k=-\infty}^{\infty} y_0(t - kT_0)}_{\text{to take care of the re-occurrence at other intervals ☺}} = \boxed{\sum_{k=-\infty}^{\infty} y_0(t - 4k) = \sum_{k=-\infty}^{\infty} (|t - 4k| - 2) \cdot 1(|t - 4k| < 2)}$

12 Let's do some Extra Problem

EXTRA

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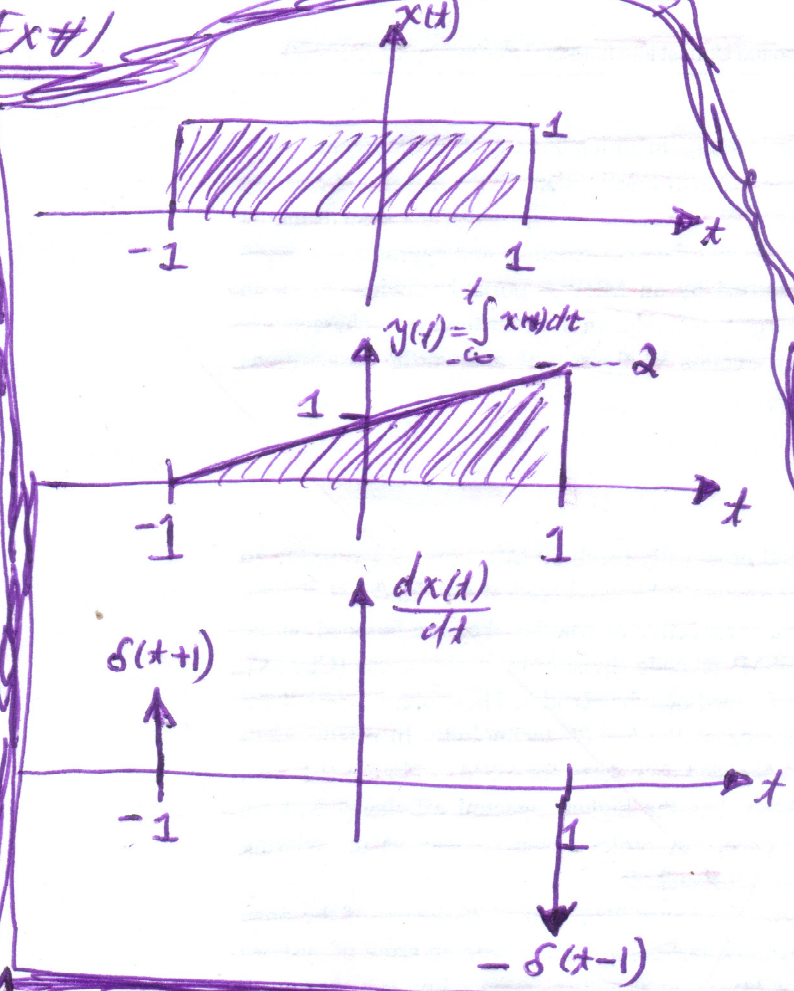
Jun. 12. 2018

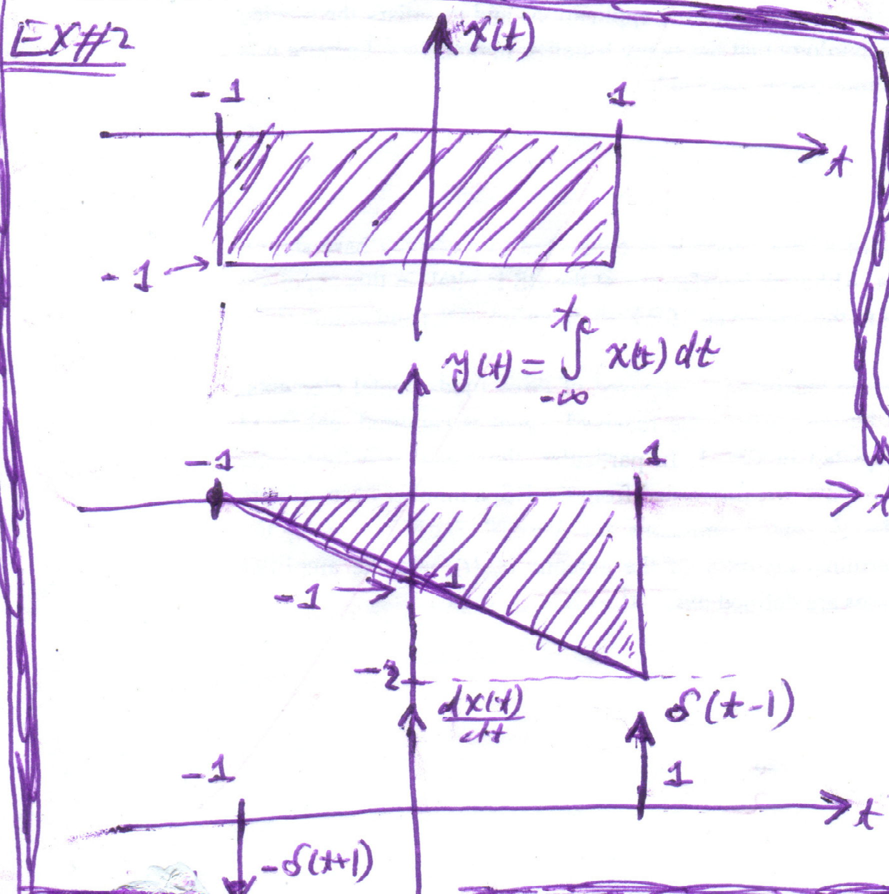
Ex#1



$$\boxed{y(t)} = \int_{-1}^{t} (1)\, dt = [t]_{-1}^{t} = \boxed{t+1}$$

$$\boxed{-1 \le t \le 1}$$

Ex#2



$$\boxed{y(t)} = \int_{-1}^{t} (-1)\, dt = -[t]_{-1}^{t}$$

$$= [t]_{t}^{-1} = -1 - t$$

$$= \boxed{-(t+1)}$$

$$\boxed{-1 \le t \le 1}$$

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EXTRA

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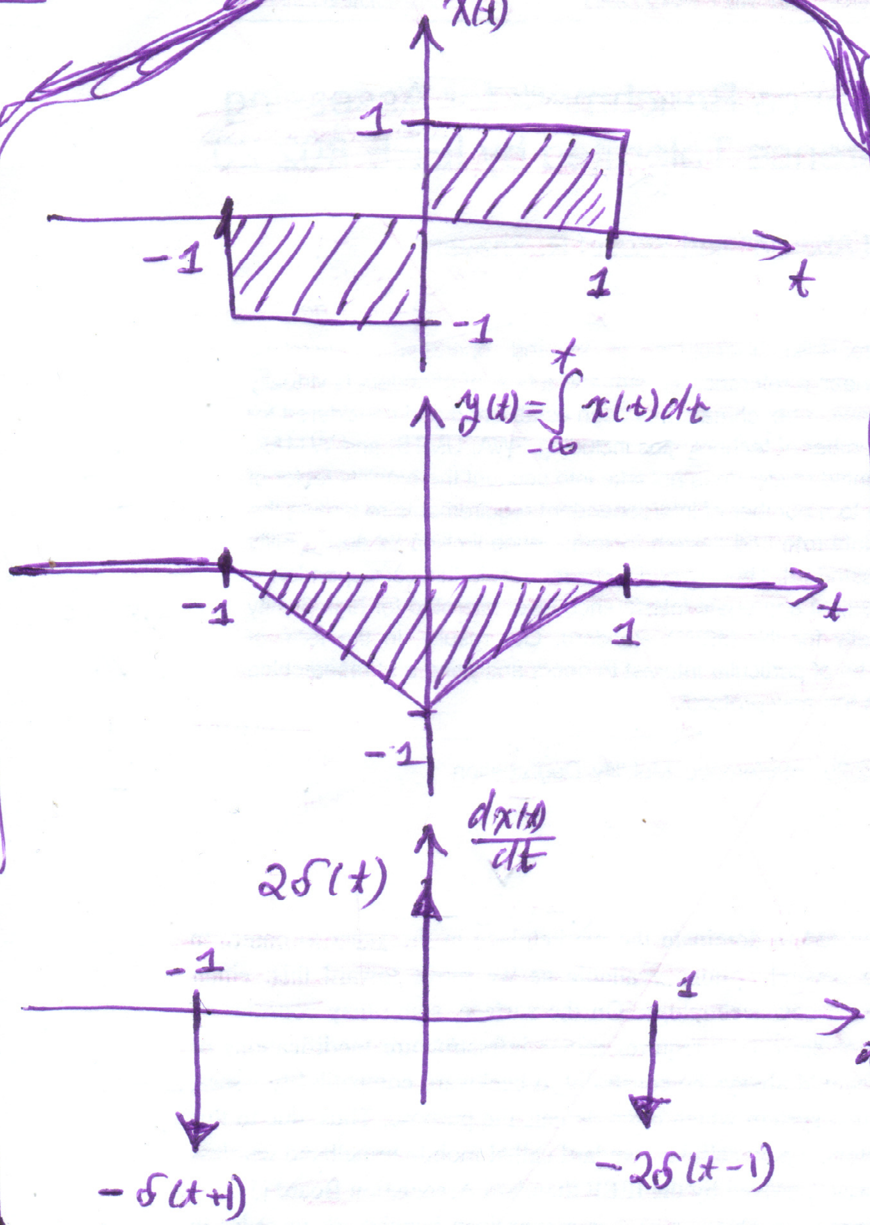
EX#3



$$y(t) = \begin{cases} \int_{-1}^{t} (-1)\,dt = -[t]_{-1}^{t} = [t]_{t}^{-1} = -1 - t = \boxed{-(t+1)} & \boxed{-1 \le t \le 0} \\ \int_{-1}^{t} x(t)\,dt = \int_{-1}^{0} (-1)\,dt + \int_{0}^{t} 1\,dt & \end{cases}$$

$$= (-1)[t]_{-1}^{0} + [t]_{0}^{t}$$

$$= [t]_{0}^{-1} + (t$$

$$= \boxed{t-1} \qquad \boxed{0 \le t \le 1}$$

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Extra

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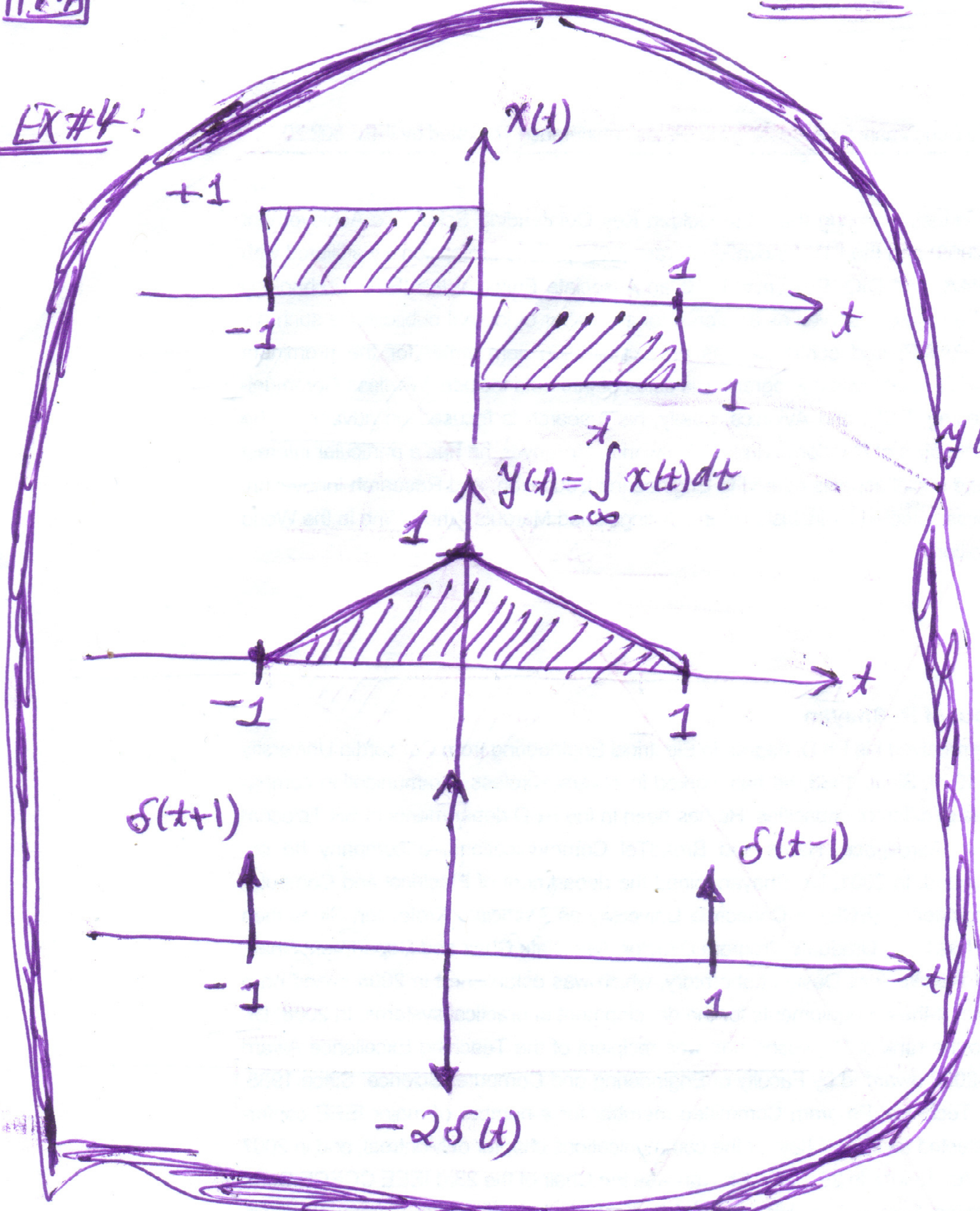
EX#4:



$$y(t) = \begin{cases} \int_{-1}^{t} (1)\,dt = [t]_{-1}^{t} = \boxed{t+1} & -1 \le t \le 0 \\[2ex] \int_{-1}^{t} x(t)\,dt = \int_{-1}^{0} (1)\,dt + \int_{0}^{t} (-1)\,dt & \end{cases}$$

$$= [t]_{-1}^{0} + (-1)[t]_{0}^{t}$$

$$= [t]_{-1}^{0} + [t]_{t}^{0}$$

$$= 0 + 1 + 0 - t$$

$$= \boxed{1-t} \qquad 0 \le t \le 1$$

The Above Shows 4 interesting Examples ☺

$\boxed{15}$

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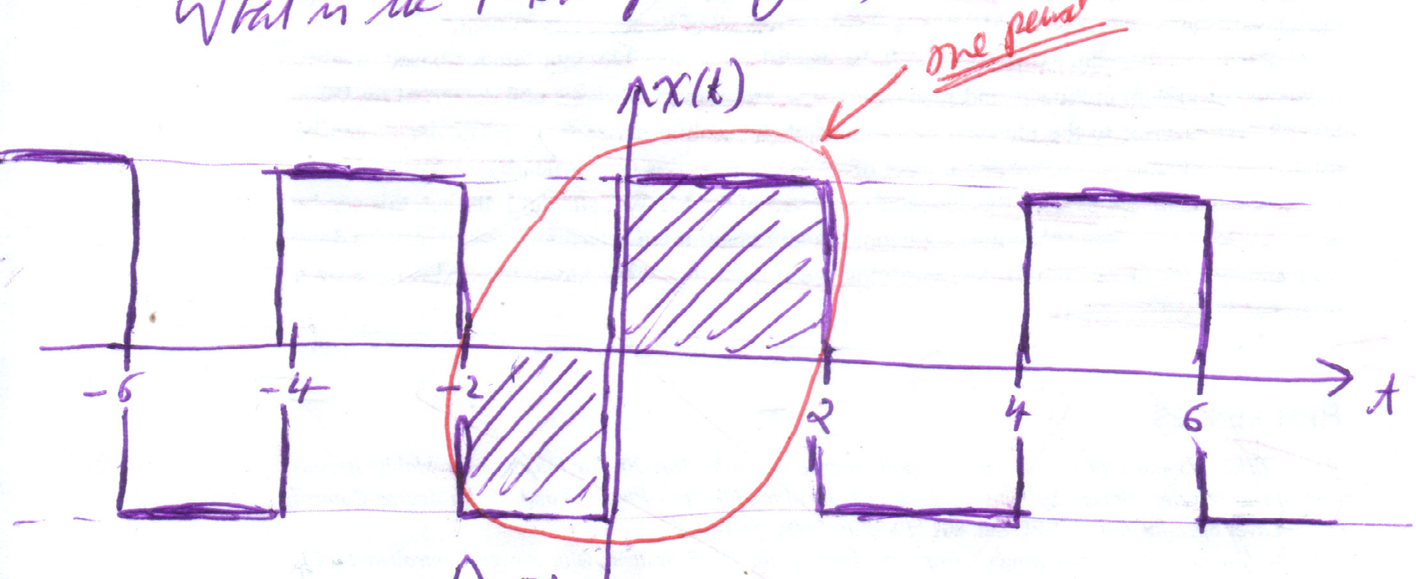
So far We answered a, b, c

Let's move to (d) $\longrightarrow$

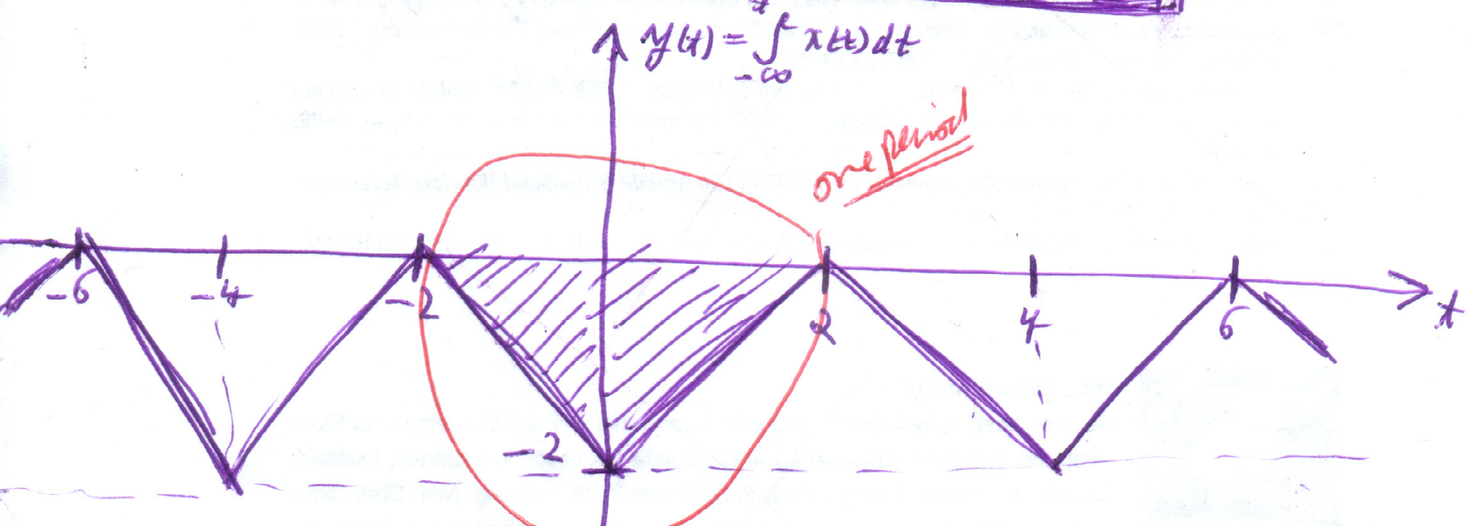
What is the F.S. of $y(t)$?



F.S.

$$\boxed{\begin{aligned} a_0 &= 0 \\ a_k &= 2(-j)^k \frac{\sin(\pi k/2)}{\pi k} \end{aligned}}$$

p. 4



$$\boxed{\omega_0} = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

$$y = \int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{\text{F.S.}} b_k = a_k \cdot \frac{1}{jk\omega_0}$$

$$\therefore \quad \boxed{b_k = \left\{ 2(-j)^k \frac{\sin(\pi k/2)}{\pi k} \; \frac{1}{(jk)} \; \frac{2}{\pi} \right\}}$$

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$$b_K = \left(\frac{2}{\pi K}\right)^2 \underbrace{(-1)^K (j)^K (-j)}_{\underbrace{(-1)^{K+1}(j)^{K+1}}_{(-j)^{K+1}}} \sin(\pi K/2)$$

$$\boxed{b_0} = \frac{\text{Area under } y(t) \text{ over } T_0}{T_0} = \frac{-\left[\frac{(4\times 2)}{2}\right]}{4} = \frac{-4}{4} = \boxed{-1}$$

$$\therefore \boxed{b_K = \left(\frac{2}{\pi K}\right)^2 (-j)^{K+1} \sin(\pi K/2)}$$

Answer to (d)

Sketch (e)

$$(-j)^{-2} = \left(\frac{1}{j}\right)^{-2} = (j)^2 = -1$$

$$K=-3 \rightarrow \boxed{b_{-3}} = \left(\frac{2}{\pi(-3)}\right)^2 (-j)^{-2} \underbrace{\sin(-3\pi/2)}_{1}$$

$$= \frac{4}{\pi^2 9} (-1)\ 1$$

$$= \boxed{\left(\frac{-4}{9\pi^2}\right)}$$

$$K=-2 \rightarrow \boxed{b_{-2}} = \left(\frac{-2}{\pi 2}\right)^2 (-j)^{-1} \underbrace{\sin(-\pi)}_{0} = \boxed{0}$$

$$K=-1 \rightarrow \boxed{b_{-1}} = \left(\frac{-2}{\pi}\right)^2 (-j)^0 \sin(-\pi/2) = \frac{4}{\pi^2} \cdot (-1) = \boxed{\frac{-4}{\pi^2}}$$

$$K=0 \rightarrow \boxed{b_0 = -1}$$

Should we work the Math for b_1, b_2, b_3? Not really!

Next page

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$y(t) \Longrightarrow$
- $\rightarrow$ Real function ✓
- $\rightarrow$ even function ✓
- $\rightarrow$ Periodic function ✓

$\therefore \boxed{a_k = a_{-k}}$

Also

$\boxed{b_1 = b_{-1} = \frac{-4}{\pi^2}} \simeq -0.4053$

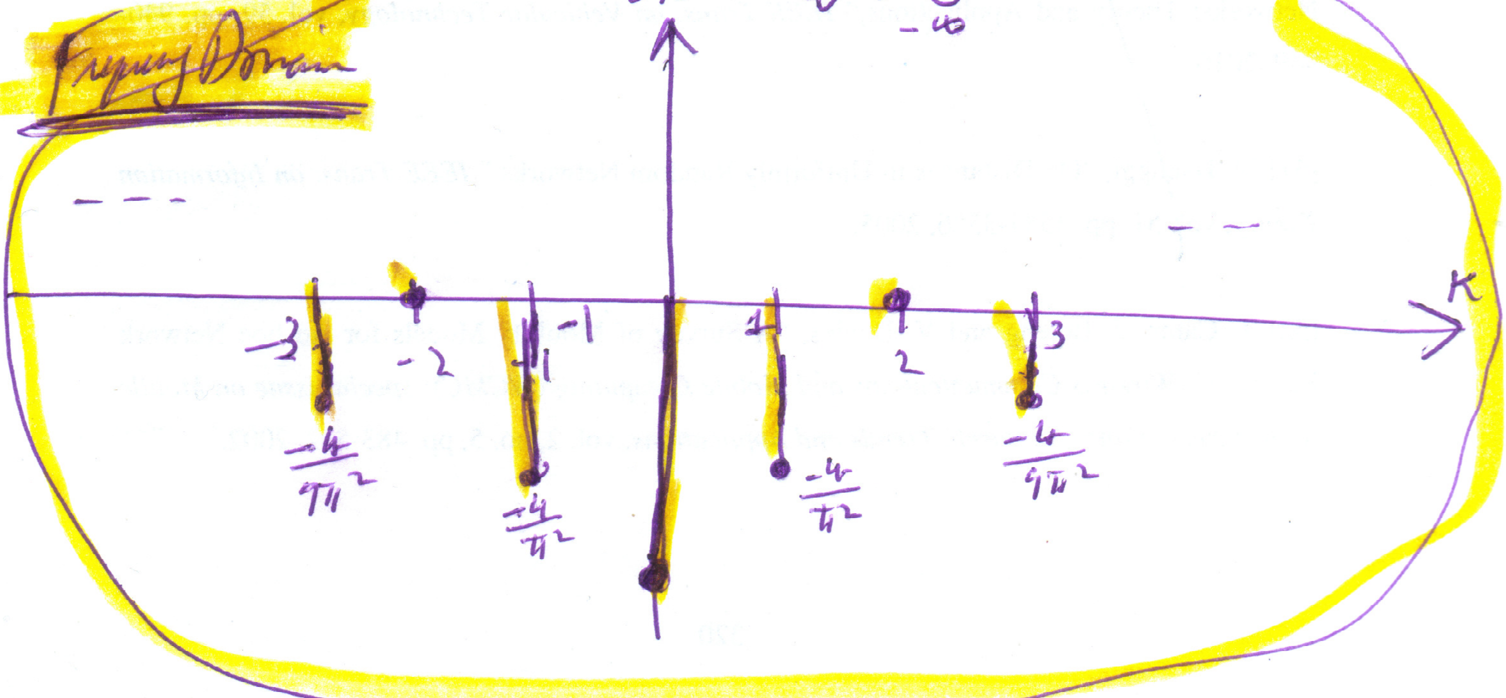
$\boxed{b_0 = -1}$

$\boxed{b_2 = b_{-2} = 0}$

$\boxed{b_3 = b_{-3} = \frac{-4}{9\pi^2}} \simeq -0.04503$

$b_k \overset{F.S.}{\longleftrightarrow} y(t) = \int_{-\infty}^{t} x(t)\,dt$

Frequency Domain



Answer!

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$$\boxed{y(t) = \int_{-\infty}^{t} x(t)\,dt}$$

F.S.

$$\boxed{\begin{aligned} b_0 &= -1 \\ b_K &= \left(\frac{2}{\pi K}\right)^2 (-j)^{K+1} \sin(\pi K/2) \end{aligned}}$$

- As Noted earlier "$y(t)$" is <u>Real</u> & <u>Even</u>

  $\therefore \quad b_K = b_{-K}$

  So let's just look at $K = 1, 2, 3, 4, 5, 6, 7, ---$

- if $K = \text{even} = 2, 4, 6, 8, 10, ---$

  $\hookrightarrow$ then $\boxed{b_K} = \left(\frac{2}{\pi K}\right)^2 (-j)^{K+1} \cancelto{0}{\sin\left(\frac{\pi K}{2}\right)}$

  $= \boxed{0}$

- if $K = \text{odd} = 1, 3, 5, 7, 9, 11, ---$

  $\hookrightarrow$ then $\boxed{b_K} = \left(\frac{2}{\pi K}\right)^2 (-j)^{K+1} \underbrace{\sin(\pi K/2)}_{1 \text{ or } -1}$

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$K = 1, 5, 9, 13, \ldots$

$$\rightarrow (-j)^{K+1} = \underbrace{(-1)^{K+1}}_{+ve} (j)^{K+1}$$

$$= (j)^{K+1}$$

$$= (j)^2, (j)^6, (j)^{10}, (j)^{14}, (j)^{18}, \ldots$$

$$= (j)^2, (j^2)^3, (j^2)^5, (j^2)^7, (j^2)^9, \ldots$$

$$= -1, -1, -1, -1, -1, \ldots$$

$$\therefore \boxed{(-j)^{K+1} = -1}$$

$$\rightarrow \sin(\pi K/2) = 1, 1, 1, 1, 1 \ldots$$

$$\therefore \boxed{\sin(\pi K/2) = 1}$$

So $\boxed{(-j)^{K+1} \cdot \sin(\pi K/2) = -1} \leftarrow K = 1, 5, 9, 13, \ldots$

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$K = 3, 7, 11, 15, \ldots$

$$\boxed{(-j)^{k+1}} = \underbrace{(-1)^{k+1}}_{+ve} (j)^{k+1}$$

$$= (j)^{k+1}$$

$$= j^4, j^8, j^{12}, j^{16}, \ldots$$

$$= (j^2)^2, (j^2)^4, (j^2)^6, (j^2)^8, \ldots$$

$$= 1, 1, 1, 1, \ldots$$

$$= \boxed{1}$$

$$\boxed{\sin(\pi k/2)} = -1, -1, -1, -1, \ldots$$

$$= \boxed{-1}$$

So $\therefore$ $\boxed{(-j)^{k+1} \sin(\pi k/2) = -1}$ $\leftarrow$ $k = 3, 7, 11, 15, \ldots$

Conclusion:

$\boxed{(-j)^{k+1} \sin(\pi k/2) = -1}$ $\leftarrow$ $\boxed{1, 3, 5, 7, 9, 11, 13, \ldots \text{ odd}}$

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$$\therefore \quad \boxed{b_K = \left(\frac{2}{\pi K}\right)^2 (-1)} \leftarrow K = 1, 3, 5, 7, 9, \ldots$$

So all, in all, we have:

alternate answer for (d) & (e)

$$\boxed{\begin{aligned} b_0 &= -1 \\ b_K &= 0 \qquad K = \text{even} = 2, 4, 6, 8, \ldots \\ b_K &= -\left(\frac{2}{\pi K}\right)^2 \qquad K = \text{odd} = 1, 3, 5, 7, \ldots \end{aligned}} \quad \boxed{b_K = b_{-K}}$$

"Nice and Easy" Representation!

(f) Evaluate: $\sum_{K=0}^{\infty} \frac{1}{(2K+1)^2} = ?$

Hint: Evaluate "$y(0)$" using "F.S."

1°

$$\underbrace{y(t)}_{\text{periodic}} \underset{\text{F.S.}}{=} \sum_{K=-\infty}^{\infty} b_K e^{jK\omega_0 t}$$

$$\underset{t=0}{y(0)} = \sum_{K=-\infty}^{\infty} b_K e^0 = \sum_{K=-\infty}^{\infty} b_K = b_0 + \underbrace{\sum_{K=-\infty}^{-1} b_K + \sum_{K=1}^{\infty} b_K}_{y(t) \text{ is real \& even } b_K = b_{-K}}$$

$$= b_0 + 2\sum_{K=1}^{\infty} b_K$$

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$$y(0) = b_0 + 2\sum_{K=1}^{\infty} b_K$$

$$= b_0 + 2\left\{\underbrace{\sum_{K=1,3,5,\dots}^{\infty} b_K}_{\text{odd}} + \underbrace{\sum_{K=2,4,6,\dots}^{\infty} b_K}_{\text{even}}\right\}$$

$$= -1 + 2\left\{\sum_{K=\text{odd}}^{\infty} b_K + 0\right\}$$

$$= -1 + 2\sum_{K=1,3,5,\dots}^{\infty} b_K$$

$$= -1 \cancel{+} 2\sum_{K=1,3,5,\dots}^{\infty} \left(\frac{2}{\pi K}\right)^2$$

$$\triangleq$$

$$= -1 - 2\cdot\frac{4}{\pi^2}\sum_{i=1,3,5,\dots}^{\infty} \frac{1}{i^2} \quad \text{map} \quad \begin{array}{l} \overset{\text{odd}}{i} = 1,3,5,7,\dots \\ K = 0,1,2,3,\dots \\ \boxed{i = 2K+1} \end{array}$$

$$\boxed{y(0) = -1 - \frac{8}{\pi^2}\sum_{K=0}^{\infty} \frac{1}{(2K+1)^2}}$$

From "Page 15" we know that $\boxed{y(0) = -2}$.

(b)

$$\therefore \quad -2 = -1 - \frac{8}{\pi^2}\sum_{K=0}^{\infty} \frac{1}{(2K+1)^2}$$

$$\triangleq \quad -1 = \frac{-8}{\pi^2}\sum_{K=0}^{\infty} \frac{1}{(2K+1)^2} \quad \triangleq \quad \boxed{\sum_{K=0}^{\infty} \frac{1}{(2K+1)^2} = \frac{\pi^2}{8}}$$

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8 Parseval's theory for F.S.

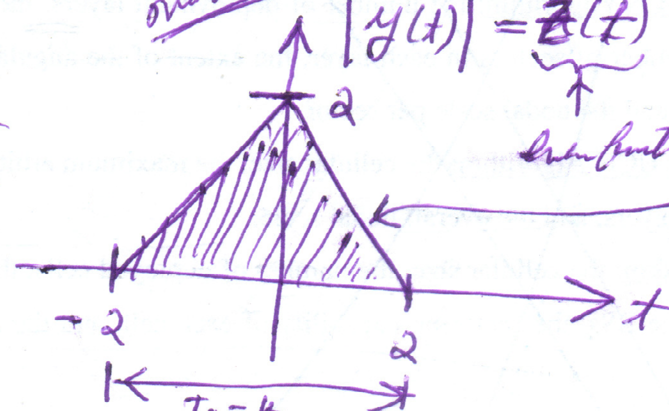
$$\underbrace{\frac{1}{T_0}\int_{T_0} |x(t)|^2 \, dt}_{LHS} = \underbrace{\sum_{k=-\infty}^{\infty} |a_k|^2}_{RHS}$$

1° LHS =

from p. 15

over one period

$$|y(t)| = \underbrace{z(t)}_{\text{even function}}$$



$y = mx + b.$

$\boxed{m} = \frac{-2}{2} = \boxed{-1}$

$\boxed{b = 2}$

$\boxed{|y(t)| = -t + 2}$

$\boxed{= 2 - t}$

$$|y(t)|^2 = \underbrace{|y(t)|}_{\text{even}} \cdot \underbrace{|y(t)|}_{\text{even}} = \text{Even} \times \text{Even} = \text{Even.}$$

$$\boxed{\int_{T_0} \text{even} = 2 \int_{T_0/2} \text{even}}$$

$\therefore$

$$LHS = \frac{1}{4} \cdot 2 \int_0^2 (2-t)^2 \, dt$$

$$= \frac{1}{2} \int_0^2 (2-t)^2 \, dt = \frac{1}{2} \int_0^2 (t-2)^2 \, dt$$

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$$LHS' = \frac{1}{2}\int_0^2 (t-2)^2\,dt$$

Let: $u = (t-2)$ $\qquad \int u^2\,du = \frac{u^3}{3} = \frac{(t-2)^3}{3}$

$du = dt$

$$\therefore LHS' = \frac{1}{2}\left[\frac{(t-2)^3}{3}\right]_0^2 = \frac{1}{6}\left[(t-2)^3\right]_0^2$$

$$= \frac{1}{6}\left[0^3 - (-2)^3\right] = \frac{-1}{6}(-1)\,8 = \frac{8}{6}$$

$$\boxed{LHS' = \frac{4}{3}}$$

2°

$$RHS' = \sum_{k=-\infty}^{\infty} |b_k|^2$$

$$= |b_0|^2 + 2\sum_{k=1}^{\infty} |b_k|^2$$

$$= |b_0|^2 + 2\sum_{k=\text{odd}=1,3,5,\dots}^{\infty} |b_k|^2$$

$$= 1 + 2\sum_{k=1,3,5,\dots}^{\infty}\left(\left(\frac{2}{\pi k}\right)^2\right)^2 = 1 + 2\sum_{k=1,3,5,\dots}^{\infty}\left(\frac{2}{\pi k}\right)^4$$

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$$RHS = 1 + \frac{2 \cdot 16}{\pi^4} \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k^4}$$

$$= 1 + \frac{32}{\pi^4} \sum_{i=1,3,5,\dots}^{\infty} \frac{1}{i^4}$$

$i = 1, 3, 5, 7, \dots$

map

$k = 0, 1, 2, 3, \dots$

$$= \boxed{1 + \frac{32}{\pi^4} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}}$$

$\boxed{i = 2k+1}$

$\therefore \quad LHS = RHS$

$$\frac{4}{3} = 1 + \frac{32}{\pi^4} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{1}{3} \cdot \frac{\pi^4}{32} = \frac{\pi^4}{96}$$

$$\boxed{\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}}$$

answer to a

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(h) Find F.T. of $\underline{\underline{y(t)}}$

$$y(t) \xleftrightarrow{F.S.} b_k = \begin{cases} b_0 = -1 \\ b_{even} = 0 \\ b_{odd} = -\left(\frac{2}{\pi k}\right)^2 \end{cases} \qquad \boxed{b_k = b_{-k}}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \xleftrightarrow{F.T.} Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} b_k \delta(\omega - k\omega_0)$$

$$Y(j\omega) = 2\pi \left[b_0 \delta(\omega) + \sum_{k=1,3,5,\ldots}^{\infty} b_k \delta(\omega - k\omega_0) + \sum_{k=-1,-3,-5,\ldots}^{-\infty} b_k \delta(\omega - k\omega_0) \right]$$

$$= 2\pi \left[-\delta(\omega) - \frac{4}{\pi^2} \sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k^2} \delta(\omega - k\omega_0) - \frac{4}{\pi^2} \sum_{k=-1,-3,-5,\ldots}^{-\infty} \frac{1}{k^2} \delta(\omega - k\omega_0) \right]$$

$$\boxed{\omega_0} = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

$$= -2\pi\delta(\omega) - \frac{8}{\pi} \sum_{i=1,3,5,\ldots}^{\infty} \frac{1}{i^2} \delta\left(\omega - i\frac{\pi}{2}\right) - \frac{8}{\pi} \sum_{i=-1,-3,-5,\ldots}^{-\infty} \frac{1}{i^2} \delta\left(\omega - i\frac{\pi}{2}\right)$$

$i = -1, -3, -5, -7, \ldots$

$k = -1, -2, -3, -4, \ldots$

map

$i = 1, 3, 5, 7, \ldots$

$k = 0, 1, 2, 3, \ldots$

$\boxed{i = 2k+1}$

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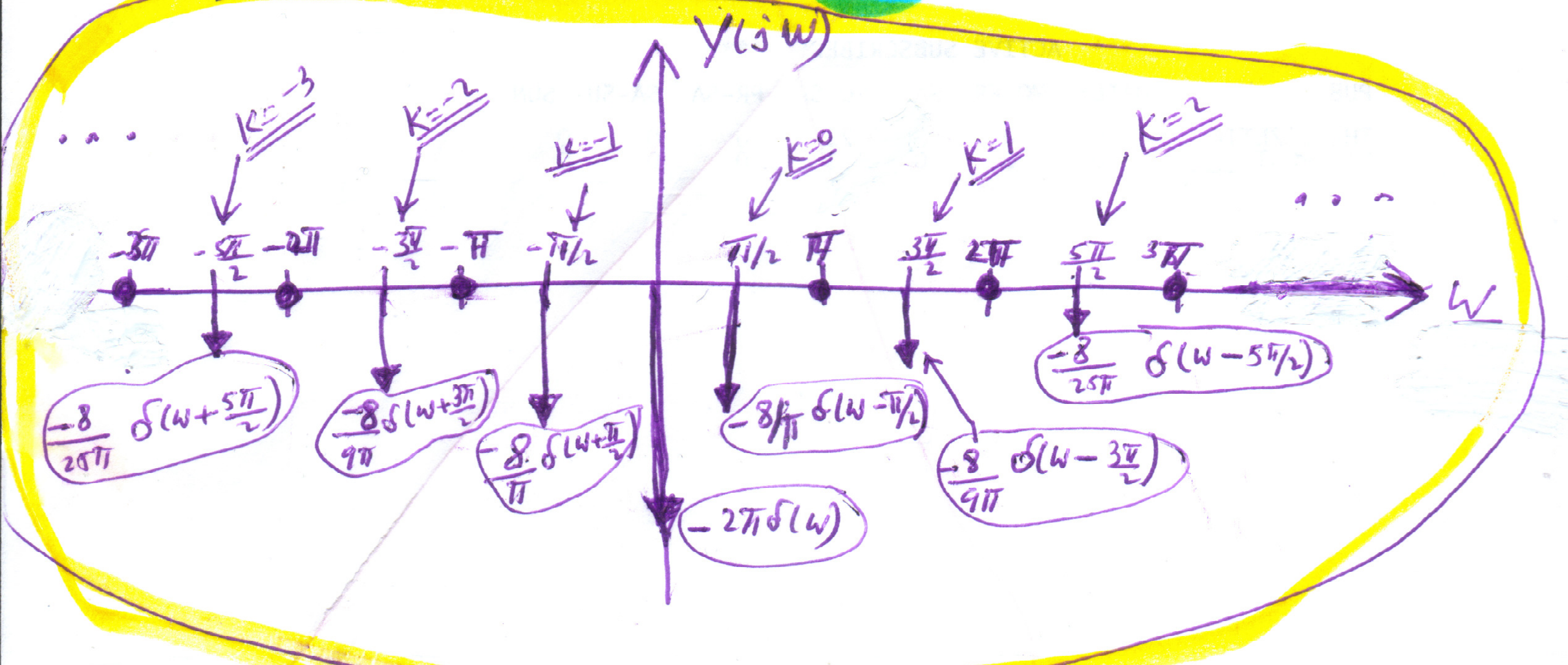
$$Y(j\omega) = -2\pi\,\delta(\omega) - \frac{8}{\pi}\sum_{K=0}^{\infty}\frac{1}{(2K+1)^2}\,\delta\!\left(\omega-(2K+1)\frac{\pi}{2}\right)$$

$$-\frac{8}{\pi}\sum_{K=-\infty}^{-1}\frac{1}{(2K+1)^2}\,\delta\!\left(\omega-(2K+1)\frac{\pi}{2}\right)$$

$$\boxed{Y(j\omega) = -2\pi\,\delta(\omega) - \frac{8}{\pi}\sum_{K=-\infty}^{\infty}\frac{1}{(2K+1)^2}\,\delta\!\left(\omega-(2K+1)\frac{\pi}{2}\right)}$$

answer (h)



F.T. of y(t)

Finally Done!