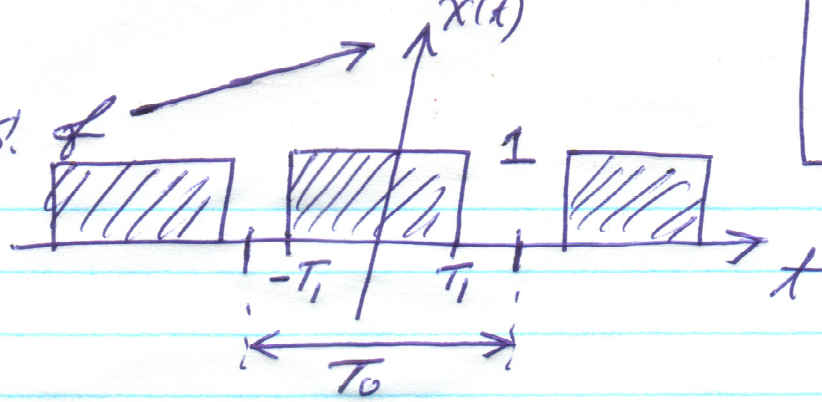
M. Abdulla
May. 31. 2013

1

Find the F.S. of ?



$\underline{1°}$ $x(t)$ is it Periodic? YES $\therefore$ F.S. exists ☺

$\underline{2°}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\,dt$$

$\underline{3°}$

$$\boxed{a_0} = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\int_{t=-T_1}^{T_1} dt = \frac{1}{T_0}\left[t\right]_{-T_1}^{T_1}$$

$$= \frac{1}{T_0}\left(T_1 + T_1\right) = \boxed{\frac{2T_1}{T_0}}$$

$\underline{4°}$

$$a_k = \frac{1}{T_0}\int_{t=-T_1}^{T_1} e^{-jk(2\pi/T_0)t}\,dt = \frac{1}{T_0}\int_{t=-T_1}^{T_1} e^{\left(\frac{-jk2\pi}{T_0}\right)t}\,dt$$

$$* \quad \boxed{\int e^{at}\,dt} = \int e^{u}\frac{du}{a} = \frac{1}{a}\int e^{u}du = \frac{e^{u}}{a} = \boxed{\frac{e^{at}}{a}}$$

$u = at \quad du = a\,dt$

$\boxed{2}$

M. Abdulla
May. 31. 2013

$$\frac{1}{j} = \frac{1}{j} \cdot \frac{j}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$$

$$-j = -j \cdot \frac{j}{j} = \frac{-j^2}{j} = \frac{+1}{j}$$

$$\boxed{a_k} = \frac{1}{T_0} \left[\frac{e^{\left(-jk\frac{2\pi}{T_0}\right)t}}{\left(-jk\frac{2\pi}{T_0}\right)} \right]_{t=-T_1}^{T_1}$$

$$= \frac{1}{\cancel{T_0}} \frac{\cancel{T_0}}{((-1)\, j\, k\, 2\pi)} \left[e^{\left(-jk\frac{2\pi}{T_0}\right)t} \right]_{-T_1}^{T_1}$$

$$= \left(\frac{j}{2\pi k}\right) \left[e^{-jk\frac{2\pi}{T_0}T_1} - e^{+jk\frac{2\pi}{T_0}T_1} \right]$$

$$= \left(\frac{-j}{2\pi K}\right) \left[e^{j\left(\frac{2\pi K T_1}{T_0}\right)} - e^{-j\left(\frac{2\pi K T_1}{T_0}\right)} \right]$$

$$= \left(\frac{1}{\pi k}\right) \left\{ \frac{\left[e^{j\left(\frac{2\pi K T_1}{T_0}\right)} - e^{-j\left(\frac{2\pi K T_1}{T_0}\right)} \right]}{2j} \right\}$$

$$= \boxed{\frac{1}{\pi K} \sin\left(\frac{2\pi K T_1}{T_0}\right)}$$

So, we got "a_0" and "a_k" of a "periodic Box" signal.

So if a signal looks like "a_k" it will probably be a modified version of a square wave signal!!!

$\boxed{3}$

Solve:

M. Abdulla
May. 31. 2013

3.23. In each of the following, we specify the Fourier series coefficients of a continuous-time signal that is periodic with period 4. Determine the signal $x(t)$ in each case.

(a) $a_k = \begin{cases} 0, & k = 0 \\ (j)^k \frac{\sin k\pi/4}{k\pi}, & \text{otherwise} \end{cases}$

(b) $a_k = (-1)^k \frac{\sin k\pi/8}{2k\pi}, \quad a_0 = \frac{1}{16}$

(c) $a_k = \begin{cases} jk, & |k| < 3 \\ 0, & \text{otherwise} \end{cases}$

(d) $a_k = \begin{cases} 1, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}$

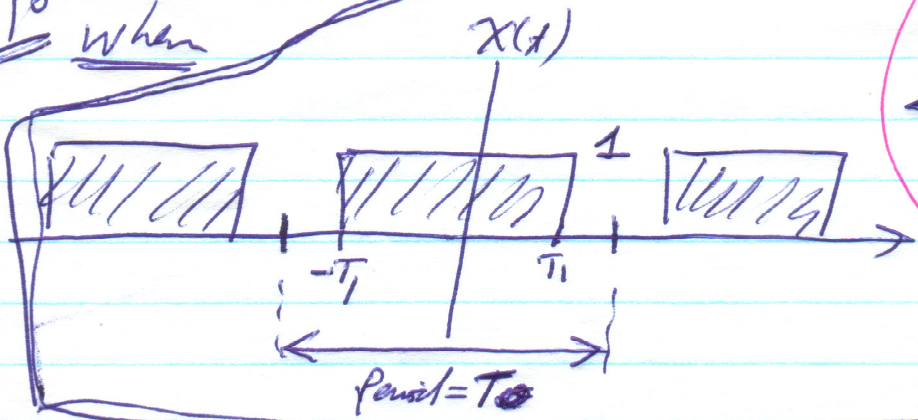
→ $x(t)$ is continuous

→ Period $= T_0 = 4$ ⟹ $\therefore \boxed{\omega_0} = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \boxed{\pi/2}$

→ What is "$x(t)$"?

Q#3.23 a) find $x(t)$ if $a_k = \begin{cases} 0 & k = 0 \\ (j)^k \frac{\sin(k\pi/4)}{(k\pi)} & k \neq 0 \end{cases}$

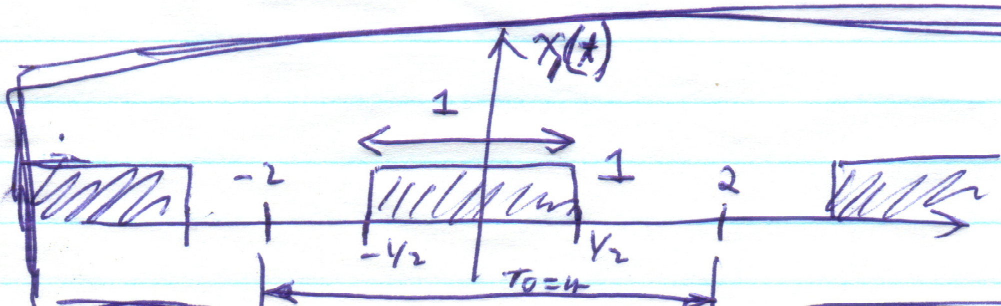
1° when



$\xleftrightarrow{F.S.}$ $a_k = \begin{cases} 2T_1/T_0 & k = 0 \\ \frac{\sin(2\pi k T_1/T_0)}{(k\pi)} & k \neq 0 \end{cases}$

Are relatively similar!

2° $\frac{2\pi k T_1}{T_0} = \frac{k\pi}{4} \quad \therefore \boxed{T_1} = \frac{T_0}{4 \cdot 2} = \frac{4}{4 \cdot 2} = \boxed{\frac{1}{2}}$



$\Longleftrightarrow$ $a_k = \begin{cases} 1/4 & k = 0 \\ \frac{\sin(\frac{\pi k}{4})}{(\pi k)} & k \neq 0 \end{cases}$

M. Abdulla
May. 31. 2013

3° Now, what should we do to remove "1/4" so that the F.S. look closer to what we want?

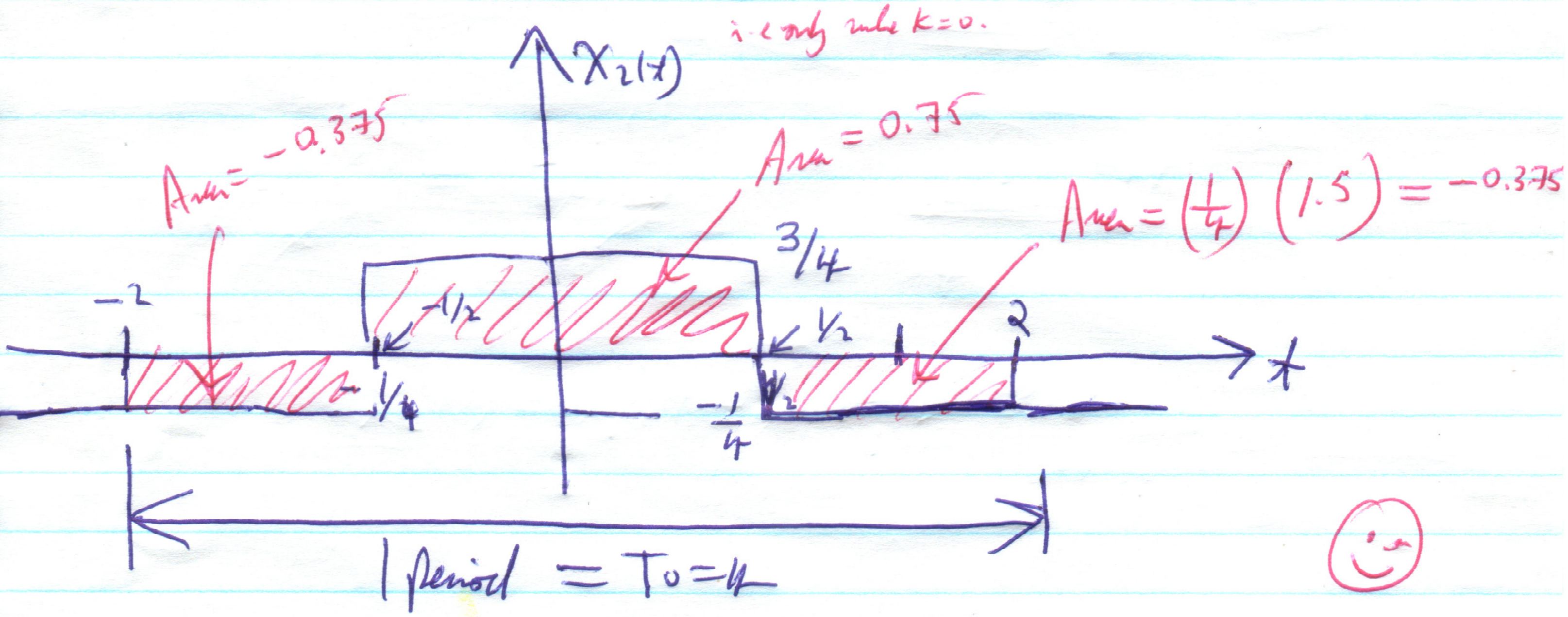
Well: $a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\cdot$ (Area under the curve)

i.e. for $k=0$

$$= \frac{1}{4}\cdot(1\times 1) = 1/4.$$

So How can we make the area under the curve over 1-period to equal "0"? $\Longrightarrow$ Just shift the graph vertically down by $1/4$ ☺

$$\therefore \boxed{x_2(t) = x_1(t) - \underbrace{1/4}_{\text{DC part only. i.e. only make } k=0.}} \xleftrightarrow{F.S.} \boxed{a_k = \begin{cases} 0 & k=0 \\ \dfrac{\sin\left(\frac{\pi k}{4}\right)}{(\pi k)} & k\neq 0 \end{cases}}$$



Total Area = 0 As expected!

$\boxed{5}$

M. Aldrick
May. 31, 2013

4°

Now, the result of 3° looks similar to what we are looking for except for $(j)^k$ part

$\boxed{(j)^k} = (e^{j\pi/2})^k = \boxed{e^{j\pi k/2}}$

We know that: $\boxed{x(t-t_0) \overset{F.S.}{\Longleftrightarrow} a_k e^{-jk\omega_0 t_0}}$

~~...~~

for $\omega_0 = \frac{\pi}{2}$

we get: $\boxed{e^{-jk\frac{\pi}{2}t_0}}$

What should "t_0" be?

$\frac{j\pi \cancel{k}}{\cancel{2}} = -j\cancel{k}\frac{\pi}{\cancel{2}} t_0 \quad \therefore \quad \boxed{t_0 = -1}$

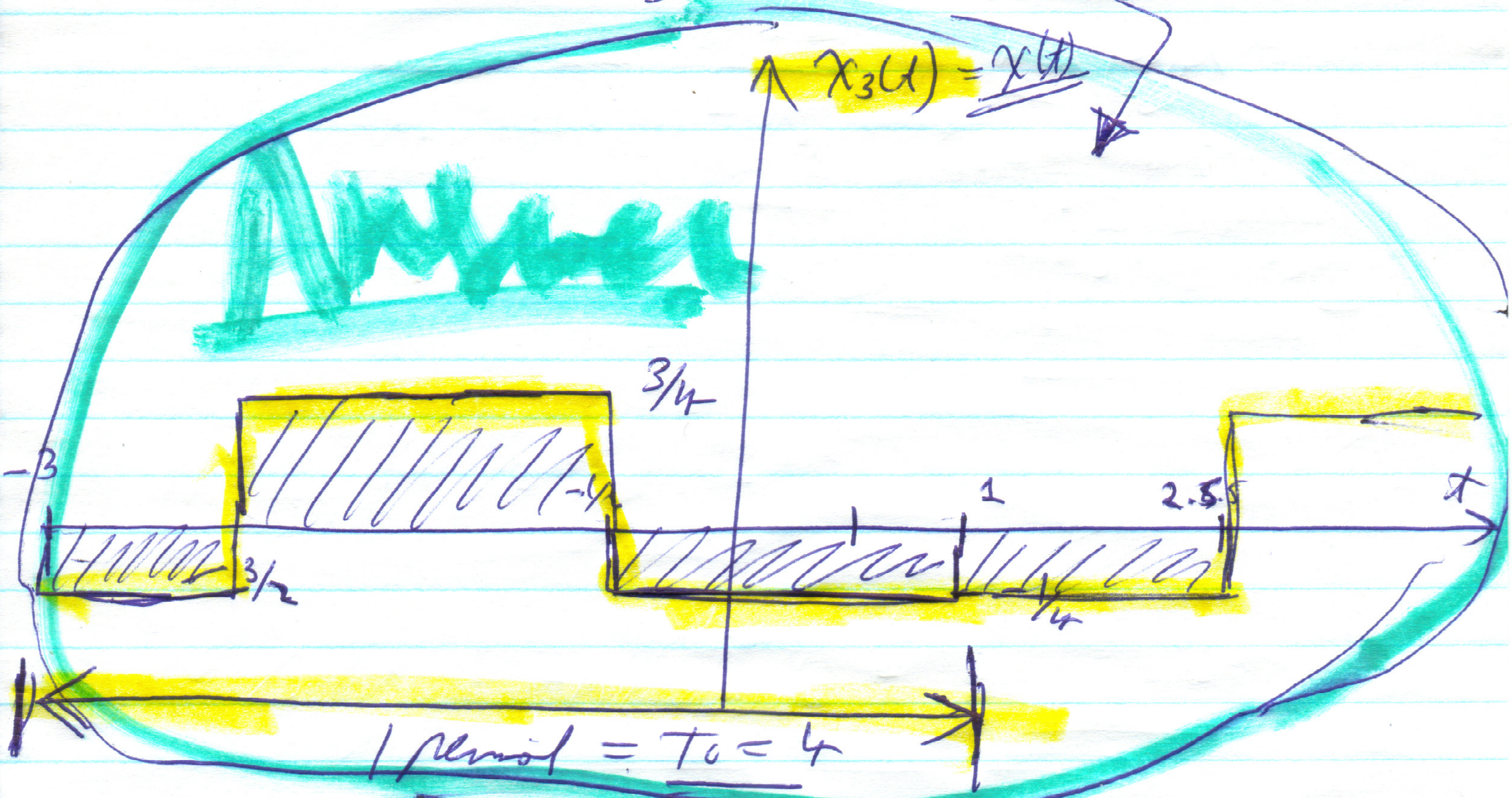
$\therefore \boxed{x(t+1) \overset{F.S.}{\Longleftrightarrow} a_k e^{jk\frac{\pi}{2}}} = \boxed{a_k (j)^k}$

M. Abdulla
May. 31. 2013

$$\boxed{x_3(t) = x_2(t+1) \Longleftrightarrow a_k = \begin{cases} 0 & k=0 \\ (j)^k \dfrac{\sin(\pi k/4)}{(\pi k)} & k \neq 0 \end{cases}}$$

We finally Got the answer!

$x_2(t+1) \Rightarrow$ i.e. take $x_2(t)$ and shift to the LEFT (←) by "1"



Very Import!

* * * *
* * * *

AC Horizontal shift ←→ $x(t \mp t_0)$ ⇒ affect "a_k" "$k \neq 0$"

DC Vertical Shift ↑↓ $x(t) \mp A$ ⇒ affect "a_0" "$k=0$"

$\boxed{7}$

M. Abdulla
May.31.2013

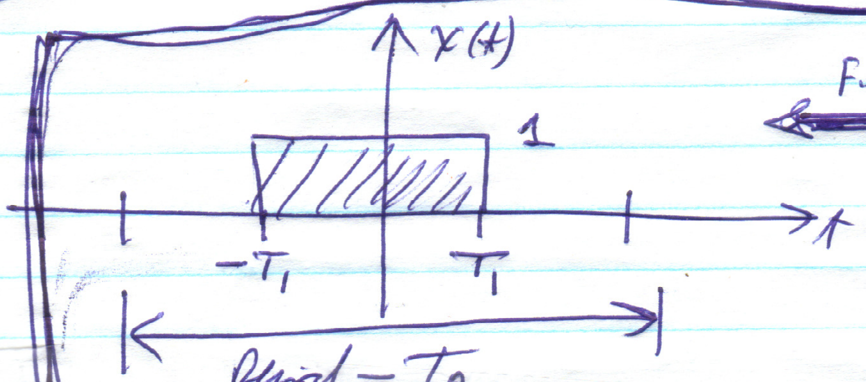
Q#3.23 b) :

$$a_k = \begin{cases} 1/16 & k=0 \\ (-1)^k \dfrac{\sin(k\pi/8)}{2k\pi} & k\neq 0 \end{cases}$$

$$\boxed{\begin{matrix} \omega_0 = \pi/2 \\ T_0 = 4 \end{matrix}}$$

get $x(t)$

1° it looks like a square signal



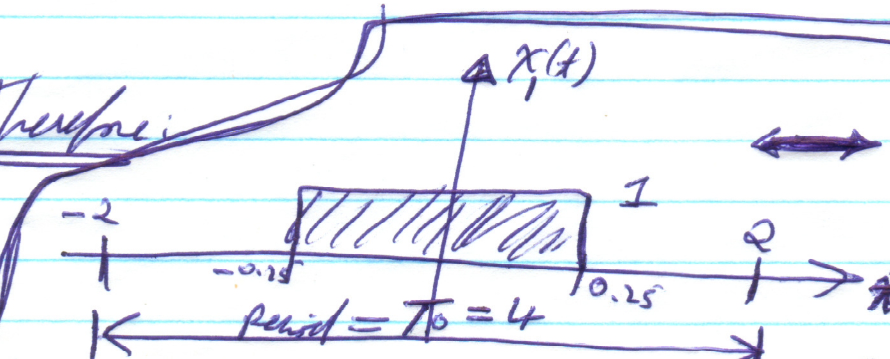
$$\xleftrightarrow{F.S.} \quad a_k = \begin{cases} (2T_1/T_0) & k=0 \\ \dfrac{\sin(2\pi k T_1/T_0)}{\pi k} & k\neq 0 \end{cases}$$

2° What do we get from the inside of the "sin"

$$\frac{2\pi k T_1}{T_0} = \frac{k\pi}{8} \Longrightarrow \boxed{T_1} = \frac{T_0}{(8)(2)} = \frac{T_0}{16} = \frac{4}{16} = \boxed{\frac{1}{4}}$$

3° $$\boxed{a_0} = \frac{2T_1}{T_0} = \frac{(2)\left(\frac{1}{4}\right)}{4} = \frac{1/2}{4} = \boxed{\frac{1}{8}}$$

4° Therefore:



$$\longleftrightarrow \quad a_k = \begin{cases} 1/8 & k=0 \\ \dfrac{\sin(k\pi/8)}{\pi k} & k\neq 0 \end{cases}$$

M. Aldulk

May. 31. 2013

5° $(-1)^k = \left(e^{j\pi}\right)^k = \boxed{e^{j\pi k}}$

$$\boxed{x(t-t_0) \xleftrightarrow{F.S.} a_k \underbrace{e^{-jk\omega_0 t_0}}_{e^{-jk\frac{\pi}{2}t_0}}}$$

What should "t_0" be?

$$-\cancel{j}k\frac{\pi}{2}\, t_0 = \cancel{j}\pi \cancel{k}$$

$$\boxed{t_0 = -2}$$

$\therefore$

$$\boxed{x_2(t) = x_1(t+2) \xleftrightarrow{F.S.} a_k = \begin{cases} 1/8 & k=0 \\ (-1)^k \dfrac{\sin(k\pi/8)}{\pi k} & k \neq 0 \end{cases}}$$

Shift to the left $\leftarrow$ by "2"

6° And lastly we need to do a scale:

$$\boxed{x_3(t) = \underbrace{\frac{1}{2}}_{\text{scale}} x_2(t) \xleftrightarrow{F.S.} \frac{1}{2} a_k = \begin{cases} 1/16 & k=0 \\ (-1)^k \dfrac{\sin(k\pi/8)}{2\pi k} & k \neq 0 \end{cases}}$$

9

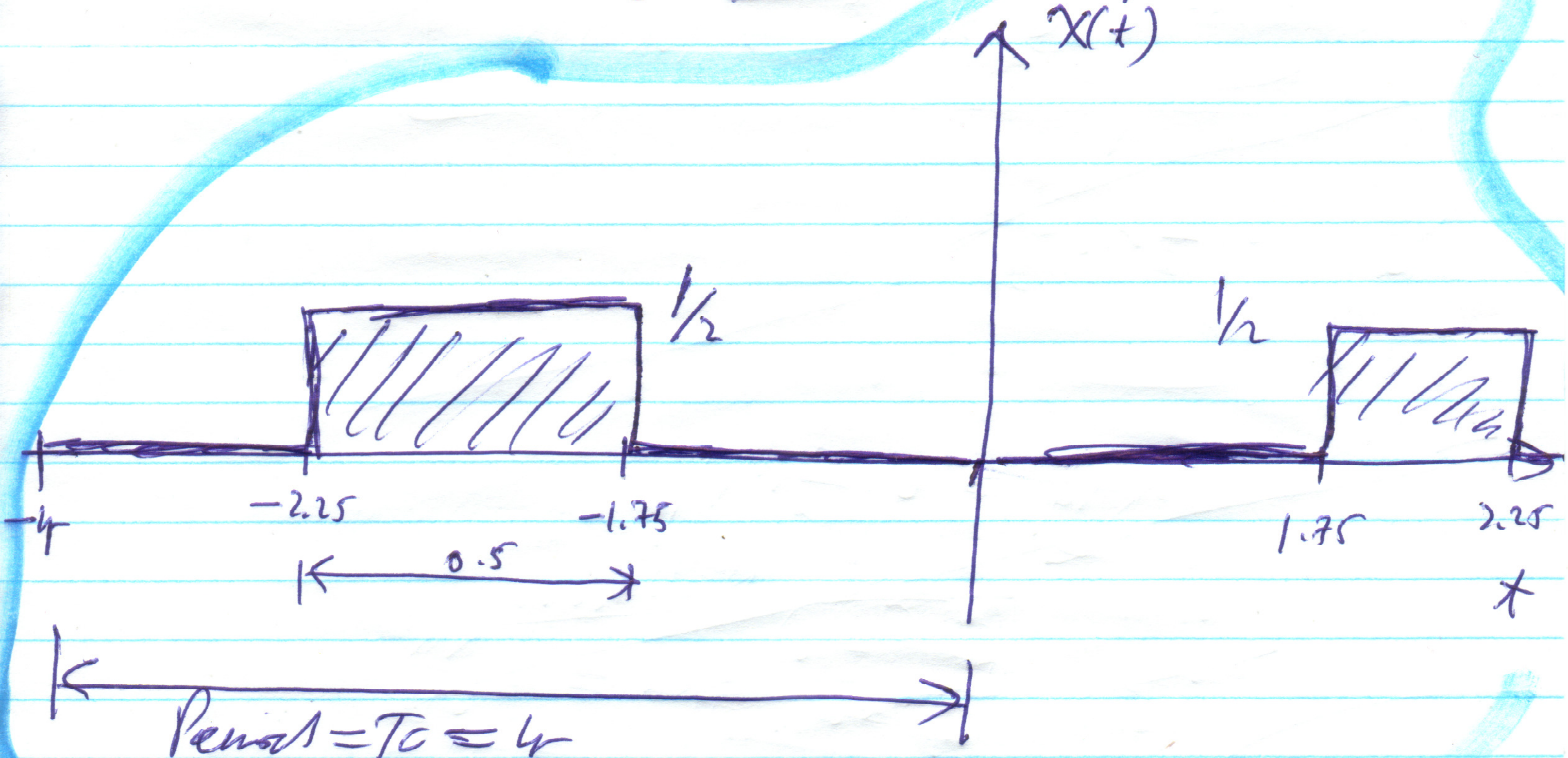
M. Abdulla
May. 31. 2013

7°

$$\boxed{x(t) = \frac{1}{2}\, x_1(t+2)}$$

Final Answer!

Final Ans!



10

M. Abdulla
May. 31. 2013

Q# 3.23 c)

$$a_K = \begin{cases} jK & |K| < 3 \\ 0 & \text{otherwise} \end{cases}$$

$T_0 = 4 \leftarrow$ Period

$w_0 = \pi/2$

$x(t) = ?$

1° What does $|K| < 3$ mean?

if $K > 0$ $\quad K < 3$

if $K < 0$ $\quad -K < 3$ or $K > -3$

$\therefore \boxed{-3 < K < 3}$

i.e.: $\boxed{K = -2, -1, 0, 1, 2}$

2° $$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jKw_0 t}$$

$$\boxed{x(t) = \sum_{K=-2,-1,0,1,2} a_K e^{jK\frac{\pi}{2}t}}$$

3° $$x(t) = \sum_{K=-2,-1,0,1,2} (jK) e^{jK\frac{\pi}{2}t}$$

$$= -2j\, e^{j\frac{\pi}{2}(-2)t} - j\, e^{-j\frac{\pi}{2}t} + 0 \cdot e^{0} + j\, e^{j\frac{\pi}{2}t} + 2j\, e^{j\frac{2\pi}{2}t}$$

$$= \boxed{-2j\, e^{-j\pi t} + 2j\, e^{j\pi t} - j\, e^{-j\frac{\pi}{2}t} + j\, e^{j\frac{\pi}{2}t}}$$

$\boxed{11}$

M. [illegible]
May. 31. 2013

$$\boxed{x(t)} = (2j)\,\frac{\left(e^{j\pi t} - e^{-j\pi t}\right)}{2j}\,(2j)$$

$$+\; j\,\frac{\left(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}\right)}{2j}\,(2j)$$

$$= 4j^2 \sin(\pi t) + 2j^2 \sin\left(\frac{\pi}{2}t\right)$$

$$= -4 \sin(\pi t) - 2 \sin\left(\frac{\pi}{2}t\right)$$

$$= \boxed{-2\left(2\sin(\pi t) + \sin\left(\frac{\pi}{2}t\right)\right)}$$

Answer!

Q #3.23 d)

$$a_K = \begin{cases} 1 & K = \text{even} = 0, \pm 2, \pm 4, \dots \\ 2 & K = \text{odd} = \pm 1, \pm 3, \pm 5, \dots \end{cases}$$

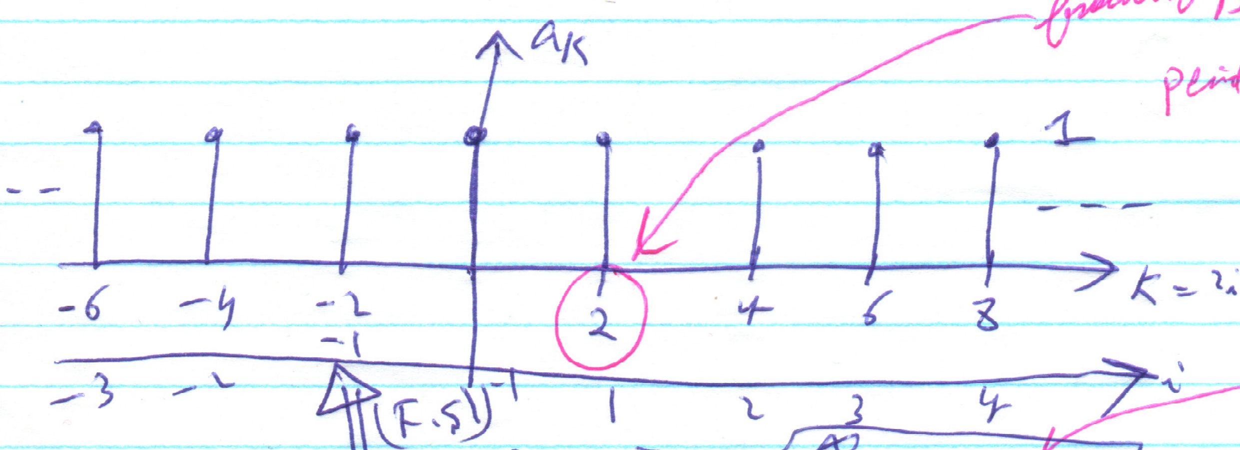
→ Find $x(t)$?

period = 4
$\omega_0 = \pi/2$

1° $\quad a_K = 1 \quad K = 0, \pm 2, \pm 4, \pm 6, \dots$

fundamental freq $= 2\omega_0 = \frac{2\pi}{\text{period}}$

period $= \frac{2\pi}{2\omega_0} = \frac{\pi}{\pi/2} = 2$

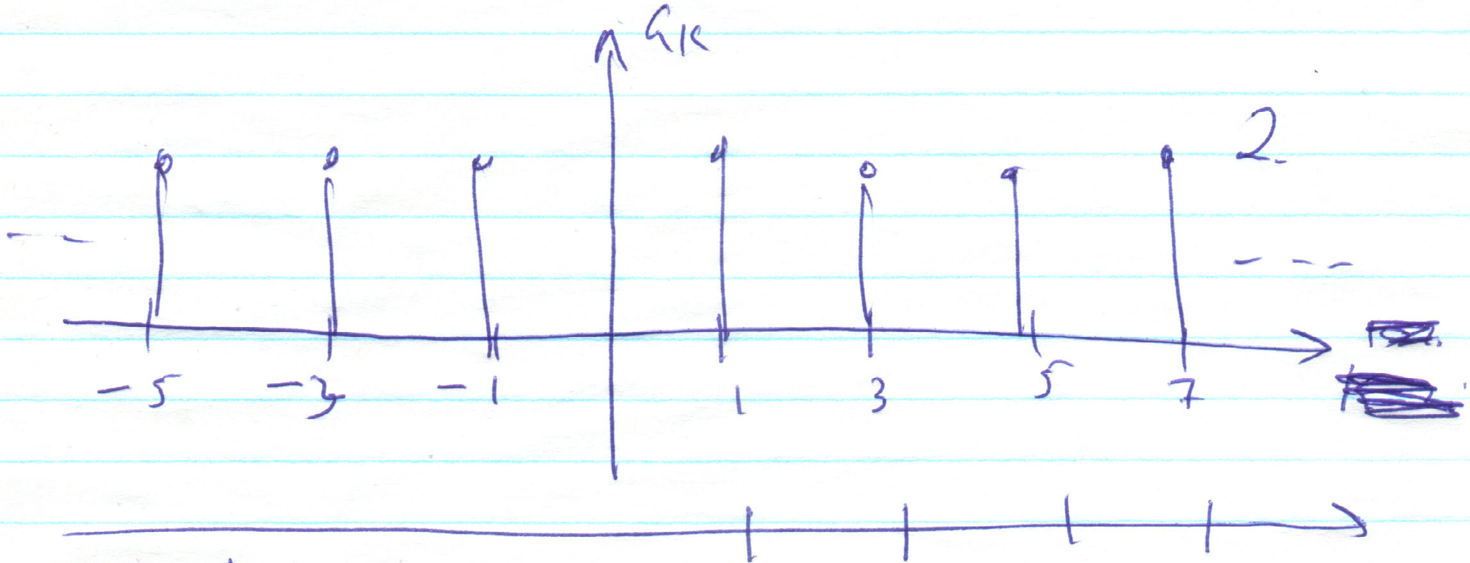


$$x_1(t) = 2\sum_{K=-\infty}^{\infty} \delta(t - 2k)$$

12

M. Alchalk
May. 31. 2013

2° $a_k = 2 \qquad k = \pm 1, \pm 3, \pm 5, \ldots$



similar to the above except the height is twice

$\therefore (\times 2)$

also, the train is shifted to right by "1"

$2 e^{j \frac{2\pi}{2}(1) t} \; x_1(t) \Longleftrightarrow 2 a_{k-1}$

$$\boxed{x_2(t) = 2 e^{j \frac{\pi}{2} t} x_1(t)}$$

$$x_2(t) = \cancel{2}4 \, e^{j \frac{\pi}{2} t} \sum_{i=-\infty}^{\infty} \delta(t - 2i)$$

$$= 4 \sum_{i=-\infty}^{\infty} e^{j \frac{\pi}{2}(2i)} \delta(t - 2i) = 4 \sum_{i=-\infty}^{\infty} (e^{j\pi})^{i} \delta(t - 2i)$$

13

M. Aldulle

May. 31. 2013

$$x(t) = x_1(t) + x_2(t)$$

$$= 2 \sum_{i=-\infty}^{\infty} \delta(t-2i) + 4 \sum_{i=-\infty}^{\infty} (-1)^i \delta(t-2i)$$

$$= 2\left[\sum_{i=-\infty}^{\infty} \delta(t-2i) + 2 \sum_{i=-\infty}^{\infty} (-1)^i \delta(t-2i) \right]$$

$$= 2\left[\sum_{i=-\infty}^{\infty} \left(\delta(t-2i) + 2(-1)^i \delta(t-2i) \right) \right]$$

$$\boxed{x(t) = 2\left[\sum_{i=-\infty}^{\infty} \delta(t-2i) \left[1 + 2(-1)^i \right] \right]}$$

final answer!