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Continuous-Time (CT) Fourier Transform (FT)

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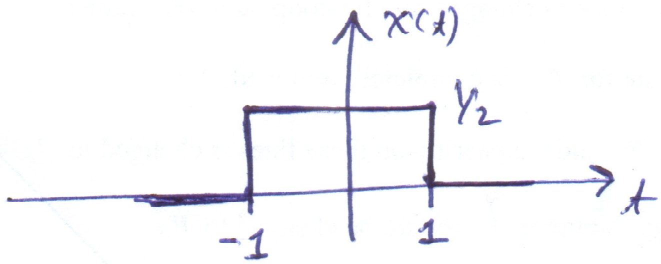
Question:

(a) Evaluate $y(t) = x(t) * x(t)$ using Convolution integral.



(b) Evaluate the Fourier Transform of $y(t) = x(t) * x(t)$

(c) Evaluate: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2(\omega)}{\omega^2}\, d\omega$ [Hint: evaluate $y(0)$ using CT-FT]

(d) Evaluate: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^4(\omega)}{\omega^4}\, d\omega$ [Hint: apply Parseval's relation to $x(t) * x(t)$]

(a)

$$x(t) \longrightarrow \boxed{x(t)} \longrightarrow \boxed{y(t) = x(t) * x(t)}$$

1^o

$$\xrightarrow{F.T.} \boxed{Y(j\omega)} = X(j\omega) \cdot X(j\omega) = \boxed{X^2(j\omega)}$$

2^o

$$y(t) = x(t) * x(t) = \int_{\tau=-\infty}^{\infty} \underbrace{x(\tau)}_{\text{Fix}}\ \underbrace{x(t-\tau)}_{\text{Move}}\, d\tau$$

3^o How to get "$x(t-\tau)$" ?

Step 1: Draw $x(\tau)$

Step 2: Get $x(-\tau)$

Step 3: shift by "t" to right $\quad x(-(\tau - t)) = x(t-\tau)$

$\boxed{2}$

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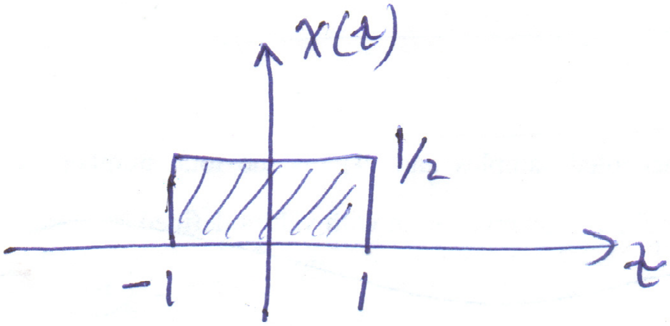
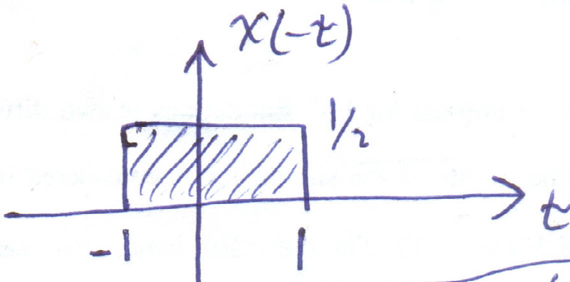
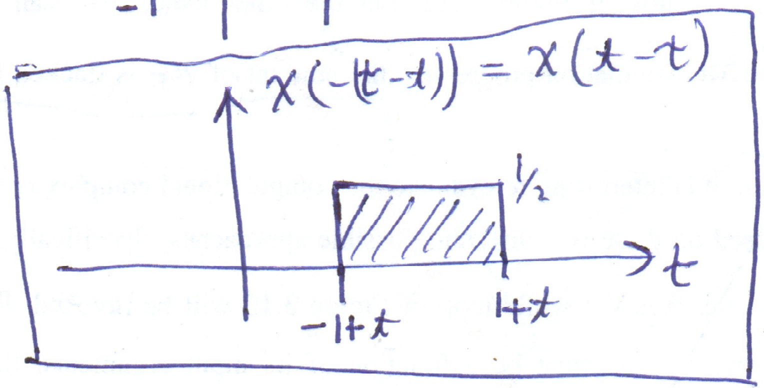
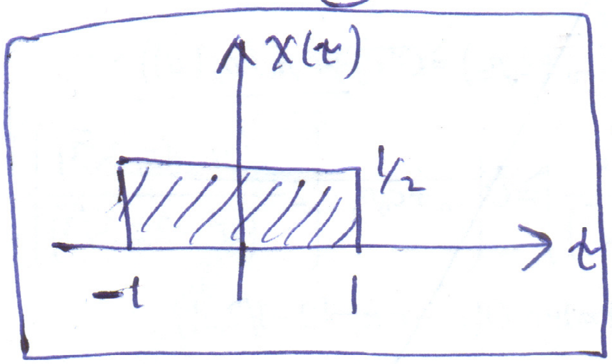






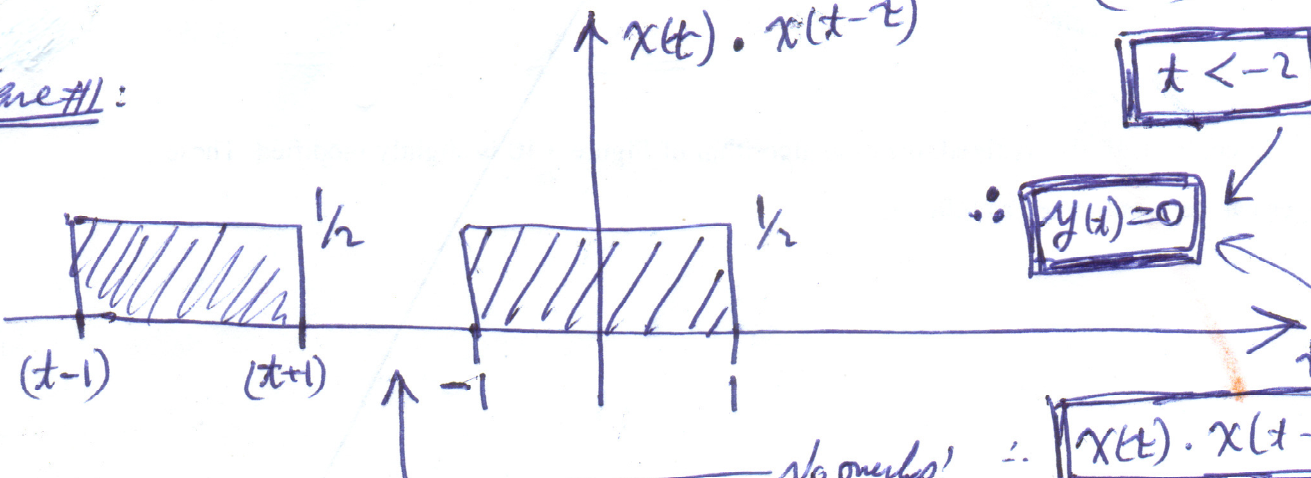
Move to different "Window"

$*$



Fix

Case #1:



$(t+1) < -1$

$\boxed{t < -2}$

$\therefore \boxed{y(t) = 0}$

$\therefore \boxed{x(\tau) \cdot x(t-\tau) = 0}$

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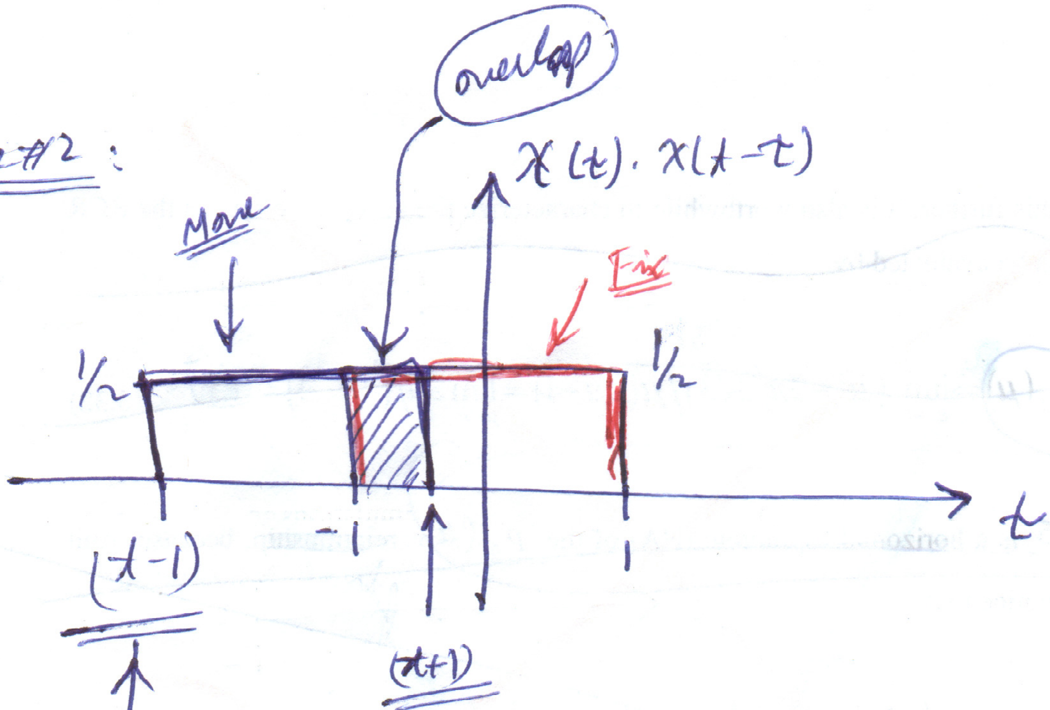
Case #2:



$(t-1) < -1$

$t < 1-1 = 0$

$\boxed{t < 0}$

$(t+1) > -1$

$\boxed{t > -2}$

$$\boxed{-2 < t < 0}$$

$$\boxed{x(\tau)\cdot x(t-\tau)} = \left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right) = \boxed{\frac{1}{4}}$$

$$\boxed{y(t)} = \int_{\tau=-1}^{\tau=(t+1)} \left(\frac{1}{4}\right) d\tau = \left(\frac{1}{4}\right) \left[\tau\right]_{\tau=-1}^{\tau=t+1}$$

$$= \frac{1}{4}\left[(t+1)+1\right] = \frac{1}{4}\left[t+2\right] = \boxed{\frac{1}{4}(t+2)}$$

$\boxed{4}$

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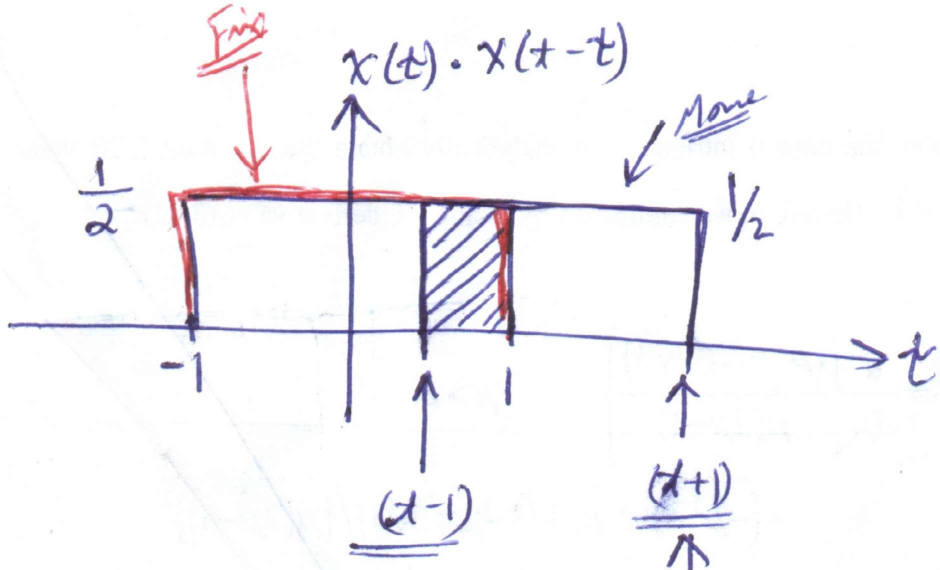
Case#3



$(t-1) < 1$

$\underline{t < 2}$

$(t-1) > -1$

$t > -1+1 = 0$

$\boxed{t > 0}$

$$\boxed{0 < t < 2}$$

$$\boxed{x(\tau)\cdot x(t-\tau)} = \left(\tfrac{1}{2}\right)\cdot\left(\tfrac{1}{2}\right) = \boxed{\tfrac{1}{4}}$$

$$\boxed{y(t)} = \int_{\tau=t-1}^{\tau=1} \left(\tfrac{1}{4}\right) d\tau = \left(\tfrac{1}{4}\right)\left[\tau\right]_{(t-1)}^{1} = \tfrac{1}{4}\left[1-(t-1)\right]$$

$$= \tfrac{1}{4}\left[1-t+1\right] = \tfrac{1}{4}\left[2-t\right] = \boxed{\frac{(2-t)}{4}}$$

$\boxed{5}$

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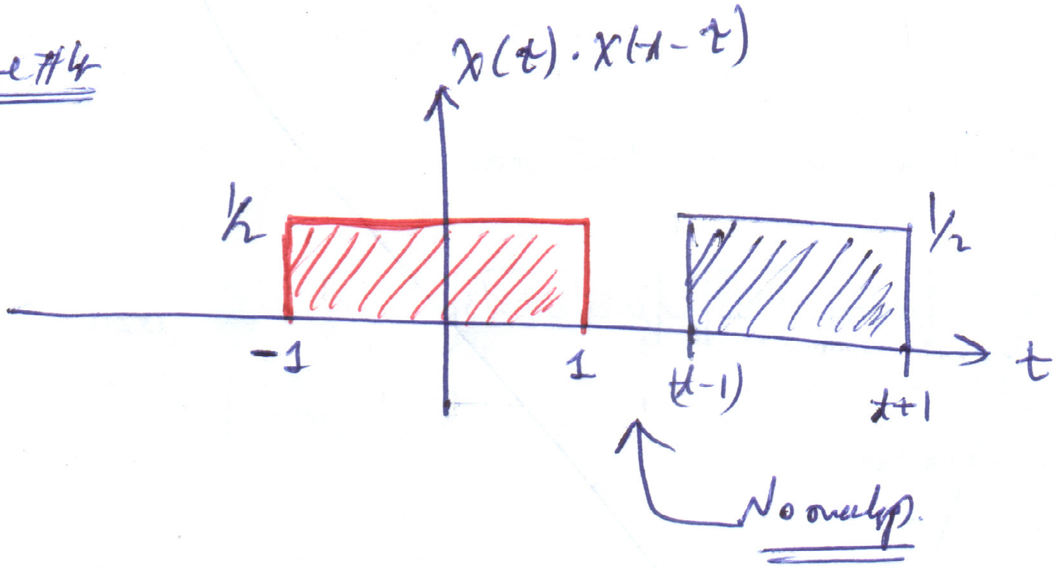
Case #4



$t-1 > 1$

$$\boxed{t > 2}$$

$$\boxed{x(\tau)\cdot x(t-\tau) = 0}$$

$$\boxed{y(t) = 0}$$

Bring Everything Together:

$$\boxed{y(t) = \begin{cases} 0 & t < -2 \\ \frac{1}{4}(t+2) & -2 < t < 0 \\ \frac{1}{4}(2-t) & 0 < t < 2 \\ 0 & t > 2 \end{cases}}$$

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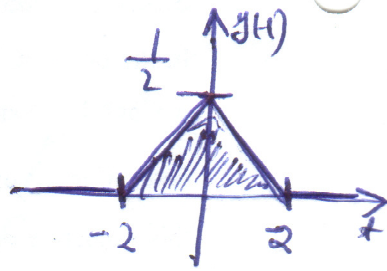
a more compact representation

$$y(t)=\begin{cases}\frac{1}{4}(2-|t|) & 0<|t|<2\\ 0 & |t|>2\end{cases}$$



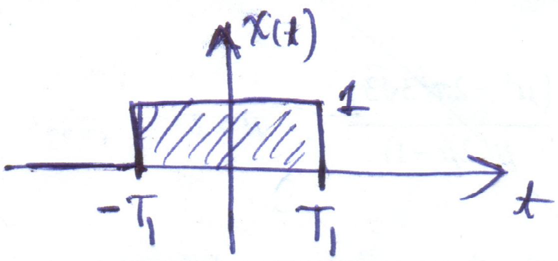
← answer to (A)

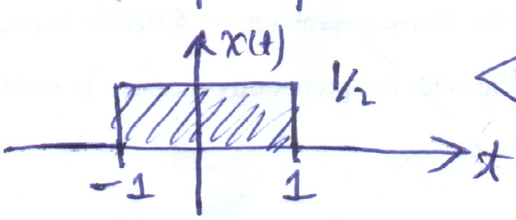
(b)

$$\boxed{Y(j\omega)=X^2(j\omega)}$$

← F.T.

$$x(t)=\begin{cases}1 & |t|<T_1\\ 0 & \text{otherwise}\end{cases} \quad \overset{\text{F.T.}}{\Longleftrightarrow} \quad X(j\omega)=\frac{2\sin(\omega T_1)}{\omega}$$





$$\overset{\text{F.T.}}{\Longleftrightarrow} X(j\omega)=\frac{\cancel{2}\sin(\omega\cdot 1)}{\omega}$$ 1/2 scale

$\therefore \boxed{T_1=1}$

$$\boxed{X(j\omega)=\frac{\sin(\omega)}{\omega}}$$

$\boxed{7}$

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$$= \boxed{Y(j\omega)} = X^2(j\omega) = \boxed{\frac{\sin^2(\omega)}{\omega^2}}$$

← answer to (B)

How to sketch?

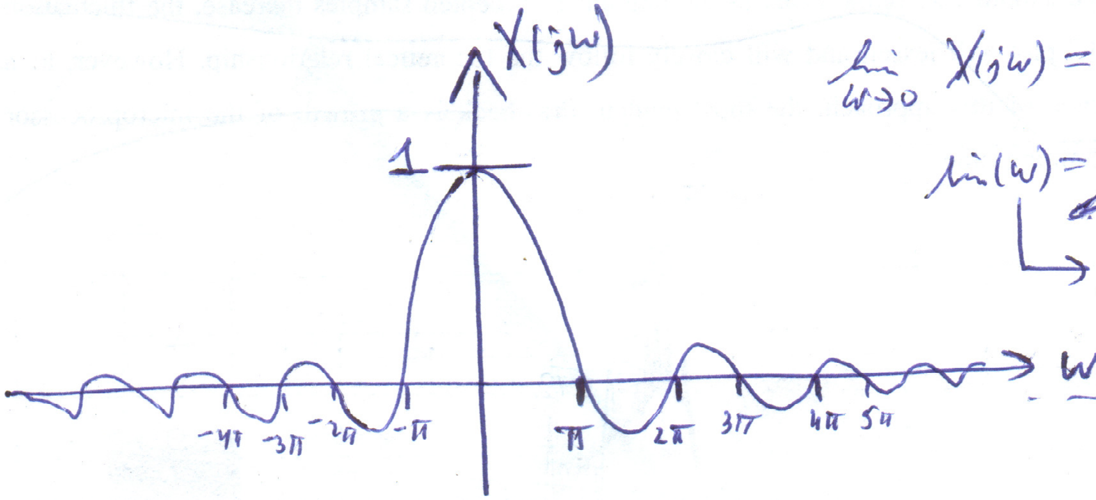
$$X(j\omega) = \frac{\sin(\omega)}{\omega} \triangleq \text{sinc}(\omega)$$

What happen at $X(j\omega)\big|_{\omega=0}$.

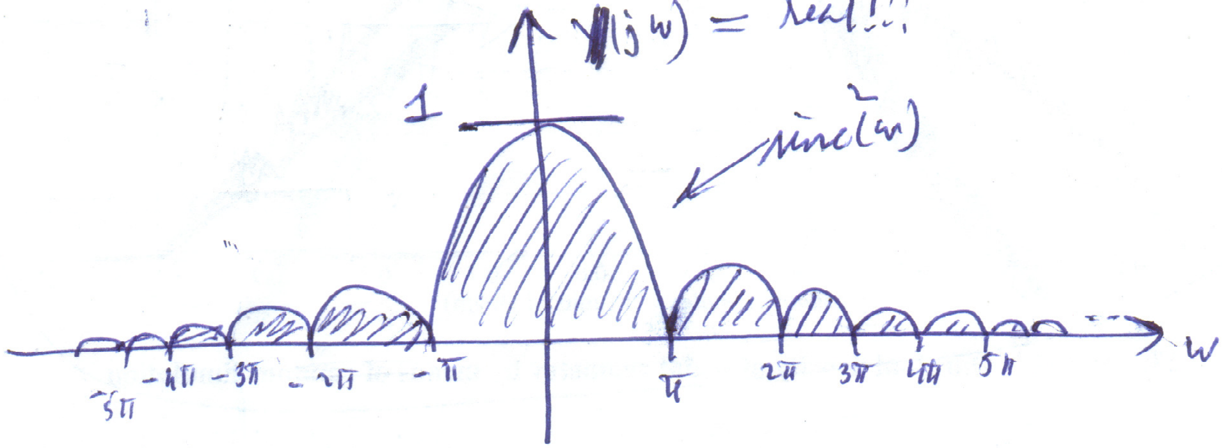
$= \frac{\sin(0)}{0}$ I.F.

$\lim_{\omega\to 0} X(j\omega) = \lim_{\omega\to 0} \frac{\cos(\omega)}{1} = \boxed{1}$

$\sin(\omega) = 0.$ → $= \pi K$ $K = \pm 1, \pm 2, \ldots$



$$Y(j\omega) = X^2(j\omega) = \frac{\sin^2(\omega)}{\omega^2} = \text{sinc}^2(\omega)$$



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Evaluate $\int_{-\infty}^{\infty} \frac{\sin^2(\omega)}{\omega^2} d\omega$ use: $y(0)$ using CT-F.T.

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(c)

1° $x(t) \xleftrightarrow{F.T.} X(j\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

2°

$$y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\sin^2(\omega)}{\omega^2}}_{\text{from part-B}} d\omega$$

from part-A

$$\boxed{\frac{1}{2} = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \frac{\sin^2(\omega)}{\omega^2}\, d\omega\right]}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin^2(\omega)}{\omega^2}\, d\omega = \pi}$$

answer to (c)

(d)

Evaluate $\int_{-\infty}^{\infty} \frac{\sin^4(\omega)}{\omega^4} d\omega$ apply Parseval to $x(t) * x(t)$

"Parseval's Relation" for "CT-FT"

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega}$$

$\boxed{9}$

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No $$\underbrace{\int_{-\infty}^{\infty} |y(t)|^2 \, dt}_{LHS} = \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega)|^2 \, d\omega}_{RHS}$$

1º $$\boxed{LHS} = \int_{-\infty}^{\infty} |y(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} y^2(t) \, dt$$

$$= \int_{-2}^{2} y^2(t) \, dt$$

$$= 2\int_{0}^{2} y^2(t) \, dt$$

$$= 2\int_{0}^{2} \frac{1}{4^2}(2-t)^2 \, dt$$

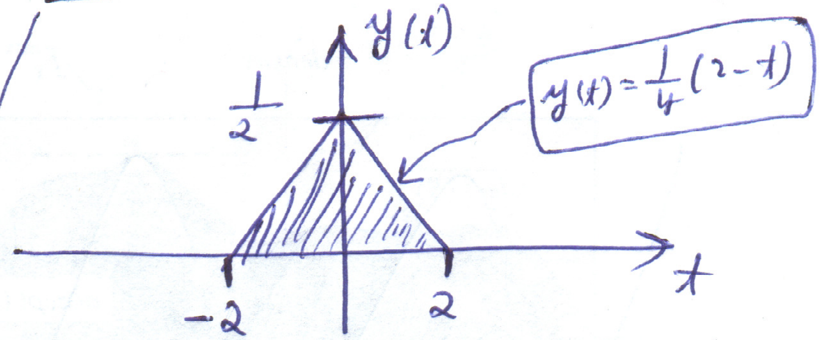
$$= \frac{2}{16} \int_{0}^{2} (t-2)^2 \, dt$$

$$= \frac{1}{8} \int_{0}^{2} (t-2)^2 \, dt$$

$$= \frac{1}{8} \left[\frac{(t-2)^3}{3} \right]_0^2$$

$$= \frac{1}{24} \left[(t-2)^3 \right]_0^2$$

$$= \frac{1}{24} \left[0 - (-2)^3 \right] = \frac{1}{24} [8] = \frac{\cancel{8}}{3 \times \cancel{8}} = \boxed{\frac{1}{3}}$$



$$|y(t)|^2 = [|y(t)|]^2 = y^2(t)$$

y(t) is +ve and real

$$= \underbrace{y(t)}_{\text{even}} \cdot \underbrace{y(t)}_{\text{even}}$$

$$= \underbrace{y^2(t)}_{\text{even}}$$

$$\boxed{\int_{T_0} \text{even} = 2\int_{T_0/2} \text{even}}$$

Let: $u = t-2$ $\quad \therefore \int u^2 \, du = \frac{u^3}{3}$

$du = dt$ $\quad = \frac{(t-2)^3}{3}$

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2° $$\boxed{RHS} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(jw)|^2 \, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{\sin^2(w)}{w^2}\right|^2 dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{\sin^2(w)}{w^2}\right]^2 dw = \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^4(w)}{w^4} \, dw}$$

3° LHS = RHS

$$\frac{1}{3} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^4(w)}{w^4} \, dw$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin^4(w)}{w^4} \, dw = \frac{2\pi}{3}}$$ ← answer

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