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Problems on Discrete-Time Fourier Transform

Question:
input: $x[n] = (1/2)^n u[n]$
output: $y[n] = (1/4)^n u[n]$

(A) Find "$H(e^{j\omega})$" and "$h[n]$"
(B) Find "$h_{in}[n]$"

(A)

1°

$$\underset{\substack{x[n] \\ X(e^{j\omega})}}{\text{in}} \longrightarrow \boxed{\begin{matrix} \text{LTI System} \\ h[n] \\ H(e^{j\omega}) \end{matrix}} \longrightarrow \underset{\substack{y[n] \\ Y(e^{j\omega})}}{\text{out}}$$

2°
$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$\therefore \boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}} \leftarrow \text{Frequency Response}$$

3°
$$x[n] = (1/2)^n u[n] \xleftrightarrow{F.T.} \boxed{X(e^{j\omega}) = \frac{1}{(1 - 1/2\, e^{-j\omega})}}$$

$\boxed{|1/2| < 1}$
yes

$$y[n] = (1/4)^n u[n] \xleftrightarrow{F.T.} \boxed{Y(e^{j\omega}) = \frac{1}{(1 - 1/4\, e^{-j\omega})}}$$

$\boxed{|1/4| < 1}$
yes

$\boxed{2}$

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4°

$$\boxed{H(e^{jw})} = \frac{Y(e^{jw})}{X(e^{jw})}$$

$$= \frac{\left[\frac{1}{(1-\frac{1}{4}e^{-jw})}\right]}{\left[\frac{1}{(1-\frac{1}{2}e^{-jw})}\right]} = \frac{1}{(1-\frac{1}{4}e^{-jw})} \frac{(1-\frac{1}{2}e^{-jw})}{1}$$

$$= \boxed{\frac{(1-\frac{1}{2}e^{-jw})}{(1-\frac{1}{4}e^{-jw})}} \longrightarrow \text{Frequency Response!}$$

5° We could actually modify $\underline{H(e^{jw})}$ so that $\underline{h[n]}$ can easily be obtained.

Say: $z = e^{jw}$

$$H(z) = \frac{(1-\frac{1}{2}z)}{(1-\frac{1}{4}z)} \begin{matrix} \leftarrow \text{order}=1 \\ \leftarrow \text{order}=1 \end{matrix} \Bigg\} \therefore$$ we can apply "long division" & simply find the "HA" (Horizontal Asymptote)

$$= A + \frac{B}{(1-\frac{1}{4}z)}$$

A: H.A.

$$HA = \lim_{z\to\infty} H(z^{-1}) = \lim_{z\to\infty} \frac{(1-\frac{1}{2}z)}{(1-\frac{1}{4}z)} \underset{\text{l'hopital Rule ☺}}{=} \lim_{z\to\infty} \frac{(-\frac{1}{2})}{(-\frac{1}{4})} = \frac{1}{2}\frac{4}{1} = \boxed{2} = \boxed{A}$$

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Estimate for "B"

Ao

$$H(z^{-1}) = 2 + \frac{B}{(1-\frac{1}{4}z)} = \frac{(1-\frac{1}{2}z)}{(1-\frac{1}{4}z)}$$

$$\frac{2(1-\frac{1}{4}z)+B}{(1-\frac{1}{4}z)} = \frac{(1-\frac{1}{2}z)}{(1-\frac{1}{4}z)}$$

$$2 - \cancel{\frac{1}{2}z} + B = 1 - \cancel{\frac{1}{2}z}$$

$$\boxed{B} = 1 - 2 = \boxed{-1}$$

$$\therefore \quad H(z^{-1}) = 2 - \frac{1}{(1-\frac{1}{4}z)}$$

A

$$\boxed{H(e^{jw}) = 2 - \frac{1}{(1-\frac{1}{4}e^{-jw})}}$$

Frequency Response

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6^o $$\boxed{h[n] = 2\delta[n] - \left(\frac{1}{4}\right)^n u[n]}$$ ← (A)

impulse Response

(B) Find "$h_{in}[n]$":

1^o $$H(e^{jw}) = \frac{(1 - 1/2\, e^{-jw})}{(1 - 1/4\, e^{-jw})}$$

$$\boxed{H_{in}(e^{jw})} = \frac{1}{H(e^{jw})} = \boxed{\frac{(1 - 1/4\, e^{-jw})}{(1 - 1/2\, e^{-jw})}}$$

2^o Let: $Z = e^{-jw}$

$$\boxed{H_{in}(z^{-1})} = \frac{(1 - 1/4\, Z)}{(1 - 1/2\, Z)} \begin{matrix} \leftarrow \text{order} = 1 \\ \leftarrow \text{order} = 1 \end{matrix} = \boxed{A + \frac{B}{(1 - 1/2\, Z)}}$$

$$HA = \lim_{z \to \infty} H_{in}(z^{-1}) = \lim_{z \to \infty} \frac{(1 - 1/4\, z)}{(1 - 1/2\, z)} = \lim_{z \to \infty} \frac{(-1/4)}{(-1/2)} = \frac{1}{4} \cdot \frac{2}{1}$$

$$\boxed{A} = \boxed{\frac{1}{2}}$$

3^o $$\left(\frac{1}{2}\right) + \frac{B}{(1 - 1/2\, Z)} = \frac{(1 - 1/4\, Z)}{(1 - 1/2\, Z)}$$

$$\frac{[(1/2)(1 - 1/2\, Z) + B]}{(1 - 1/2\, Z)} = \frac{(1 - 1/4\, Z)}{(1 - 1/2\, Z)}$$

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$$\frac{1}{2} - \cancel{\frac{1}{4}z} + B = 1 - \cancel{\frac{1}{4}z}$$

$$\boxed{B} = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

$$\therefore \boxed{H_{in}(z^{-1}) = \left(\frac{1}{2}\right) + \frac{(1/2)}{(1 - 1/2\, z)}}$$

or

$$\boxed{H_{in}(e^{jw}) = \left(\frac{1}{2}\right) + \frac{(1/2)}{(1 - 1/2\, e^{-jw})}}$$

$(F.T.)^{-1}$

$$h_{in}[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\left(\frac{1}{2}\right)^{n} u[n]$$

or

$$\boxed{h_{in}[n] = \frac{1}{2}\delta[n] + \left(\frac{1}{2}\right)^{n+1} u[n]}$$

B

answer to B

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Question:

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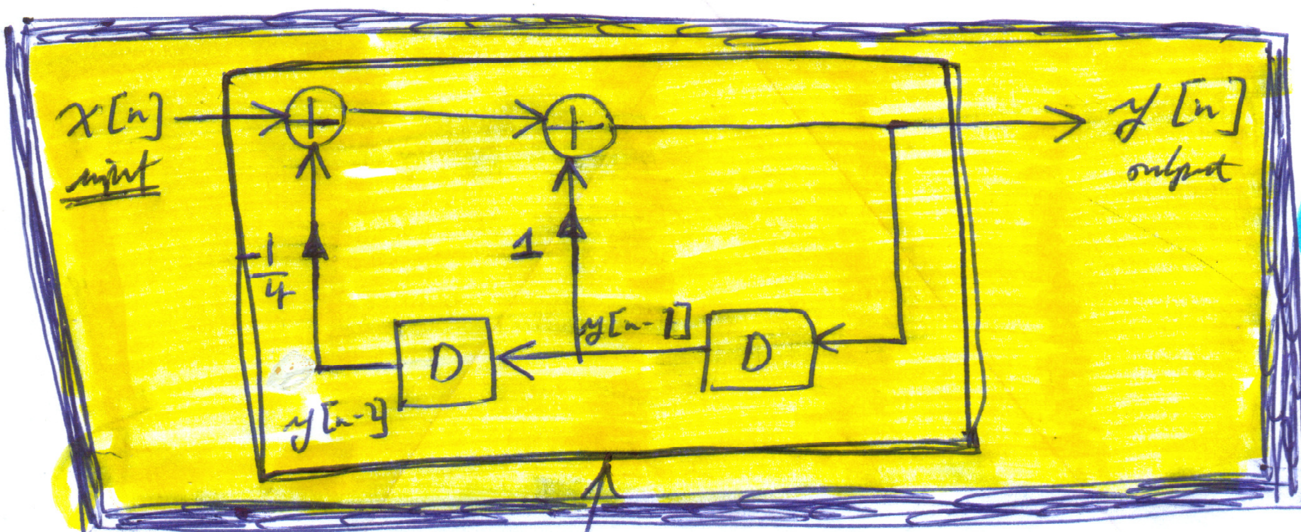
System: $\frac{1}{4}y[n-2] - y[n-1] + y[n] = x[n]$

(A) Sketch Block Diagram
(B) Find Frequency Response of the system
(C) Find Impulse Response of the System
(D) Is this system stable?

(A) To draw block diagram isolate for "y[n]"

$$\therefore \boxed{y[n] = x[n] + y[n-1] - \frac{1}{4}y[n-2]}$$

Block diagram of the system



(A)

LTI System

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(B) $\frac{1}{4} Y(e^{jw}) e^{-jw2} - Y(e^{jw}) e^{-jw} + Y(e^{jw}) = X(e^{jw})$

$$Y(e^{jw}) \left\{ \frac{1}{4} e^{-j2w} - e^{-jw} + 1 \right\} = X(e^{jw})$$

$$\boxed{H(e^{jw})} = \frac{Y(e^{jw})}{X(e^{jw})} = \boxed{\frac{1}{\frac{1}{4}(e^{-jw})^2 - (e^{-jw}) + 1}}$$

Frequency Response

Let: $z = e^{-jw}$

$$\frac{1}{4} z^2 - z + 1 = \frac{1}{4} \underbrace{(z^2 - 4z + 4)}_{\text{factor!}}$$

$a = 1$
$b = -4$
$c = 4$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - (4)(1)(4)}}{2}$$

$$= \frac{4 \pm 0}{2} = 2 \quad \therefore (z-2)^2$$

$$= \frac{1}{4}(z-2)^2$$

$$= \frac{1}{4}(2-z)^2$$

$$= \frac{1}{4}\left(2\left(1 - \frac{z}{2}\right)\right)^2$$

$$= \frac{4}{4}\left(1 - \frac{z}{2}\right)^2 = \left(1 - z/2\right)^2$$

$$= \boxed{\left(1 - \frac{1}{2} e^{-jw}\right)^2}$$

Frequency Response of the system

$$\therefore \boxed{H(e^{jw}) = \frac{1}{\left(1 - \frac{1}{2} e^{-jw}\right)^2}}$$

B

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(c)

$$\boxed{\frac{(n+r-1)!}{n!\,(r-1)!}\, a^n u[n] \xleftrightarrow{F.T.} \frac{1}{(1-ae^{-jw})^r} \qquad |a|<1}$$

$$a = \frac{1}{2}$$

$$r = 2$$

$|1/2| < 1$ Yes ☺

$\therefore$

$$h[n] = \frac{(n+2-1)!}{n!\,1!}\left(\frac{1}{2}\right)^n u[n]$$

$$= \frac{(n+1)!}{n!}\left(\frac{1}{2}\right)^n u[n]$$

$$\frac{(n+1)!}{n!} = \frac{\cancel{1.2.3.\cdots(n-1).n}\,(n+1)}{\cancel{1.2.3.\cdots(n-1)\,n}}$$

$$= (n+1)$$

$\therefore$ $$\boxed{h[n] = (n+1)\left(\frac{1}{2}\right)^n u[n]}$$

impulse Response!

Bounded In Bounded output BIBO

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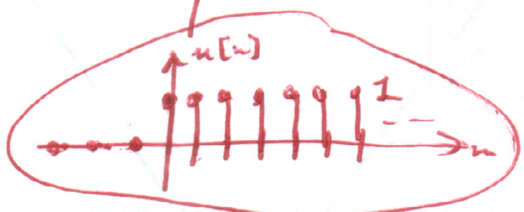
① Is the system stable?

A system is stable if $\Longrightarrow$ $\boxed{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}$

1° $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} \left|(n+1)\left(\frac{1}{2}\right)^n u[n]\right| = \sum_{n=0}^{\infty} \left|\underbrace{(n+1)}_{+ve}\underbrace{\left(\frac{1}{2}\right)^n}_{+ve}\right|$



$\therefore$ $\boxed{\sum_{n=0}^{\infty} (n+1)\left(\frac{1}{2}\right)^n \overset{?}{<} \infty}$

2° Say: $\sum_{n=0}^{\infty} (n+1)\, a^n$ $\qquad \boxed{0<a<1}$

$= \frac{d}{da}\left\{\sum_{n=0}^{\infty} a^{(n+1)}\right\} = \sum_{n=0}^{\infty} \frac{d}{da}\left\{a^{(n+1)}\right\} = \sum_{n=0}^{\infty} (n+1)\, a^n$ QED ☺

$= \frac{d}{da}\left\{a\sum_{n=0}^{\infty} a^n\right\}$ $\qquad \boxed{0<a<1}$

$\therefore$ $\boxed{\sum_{n=0}^{\infty} (n+1)\, a^n = \frac{d}{da}\left\{a\sum_{n=0}^{\infty} a^n\right\}}$

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3°

$$\boxed{\sum_{n=0}^{\infty} a^n} = \frac{a^0 - a^{\infty}}{1-a} = \frac{1-0}{(1-a)} = \boxed{\frac{1}{(1-a)}}$$

4°

$$\sum_{n=0}^{\infty} (n+1)\, a^n = \frac{d}{da}\left\{a \cdot \frac{1}{(1-a)}\right\}$$

$$= a' \cdot (1-a)^{-1} + a \cdot \left((1-a)^{-1}\right)'$$

$$= (1-a)^{-1} + a(-1)(1-a)^{-2}(-1)$$

$$= \frac{1}{(1-a)} + \frac{a}{(1-a)^2}$$

$$= \frac{(1-a)}{(1-a)^2} + \frac{a}{(1-a)^2}$$

$$= \frac{1}{(1-a)^2}$$

$$\therefore \boxed{\sum_{n=0}^{\infty} (n+1)\, a^n = \frac{1}{(1-a)^2} \qquad 0 < a < 1}$$

5° if $\boxed{a = \frac{1}{2}} \Rightarrow \boxed{\sum_{n=0}^{\infty} (n+1)\left(\frac{1}{2}\right)^n} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{1/4} = \boxed{4} < \infty$

$\boxed{\text{"}h[n]\text{" is BIBO Stable}}$

done!