Quick Review of: "LTI Systems" + "Fourier Series"

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May. 31. 2013

LTI System

* If a system is "Linear" and "Time Invariant" then it has some very very very very very interesting properties
  - i.e. LTI Systems are a GOOD Thing!! ☺

* Why are LTI so interesting?
  - ① the nice idea of $\boxed{\text{Convolution } (*)}$ exist and can be applied.
  - ② Various $\boxed{\text{"Transform"}}$ can be applied to gain insight about the "frequency domain" of a signal.

e.g.:
- Fourier Series (F.S.)
- Fourier Transform (F.T.)
- Z-Transform (Z.T.)
- Laplace Transform (L.T.)

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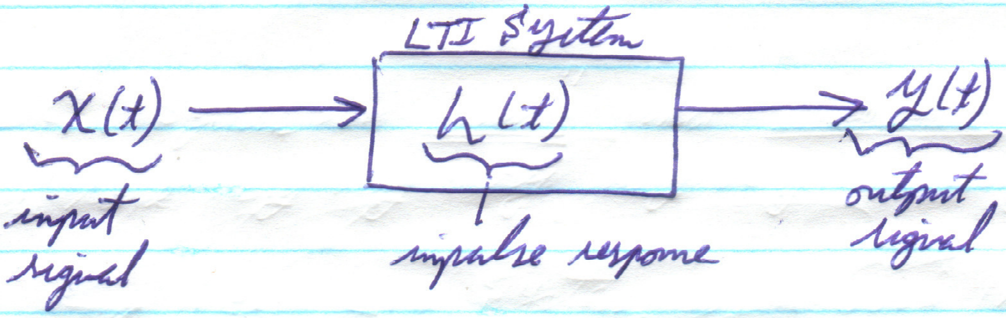
Explain the "math" of LTI System

(1) Signals that are "Continuous in Time"



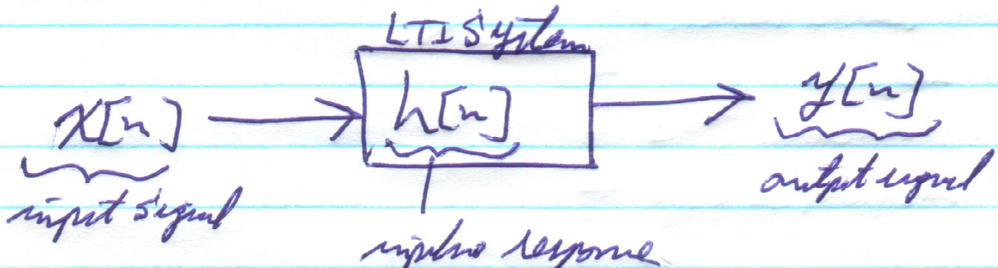
$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

"Convolution operator"

use whichever is easier to integrate! ☺

(2) Signals that are "Discrete in Time"



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

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Fourier Series of a Continuous Time Signal

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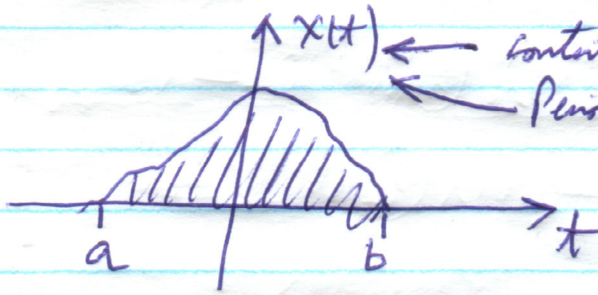
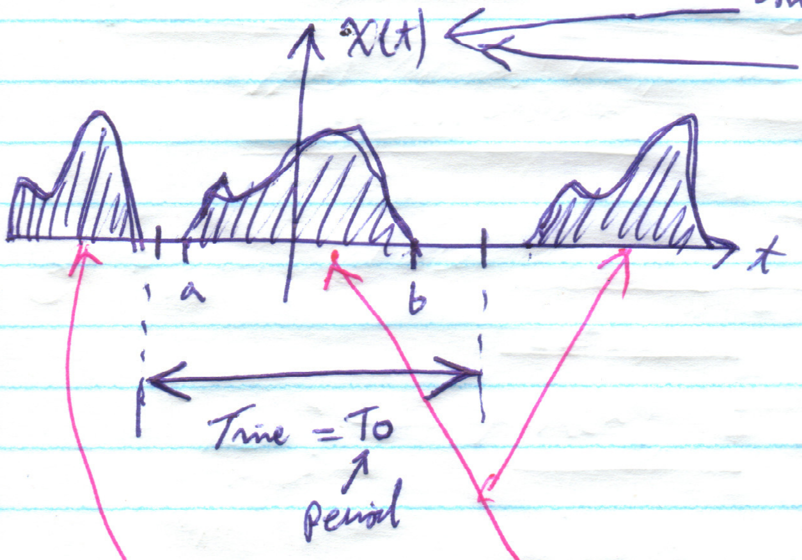
* * * Very Important!!!!!!

F.S. exists provided the signal is "PERIODIC" "in Time"

What does "periodic in Time" mean?



continuous signal: YES
Periodic in Time: NO



continuous: YES
Periodic: YES

because the same exact "signal shape" is repeated after "To" seconds

The shape should be the same exact replica

-- Sorry I am a BAD DRAWER

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• F.S. for $x(t) \rightarrow$ Periodic

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}}$$

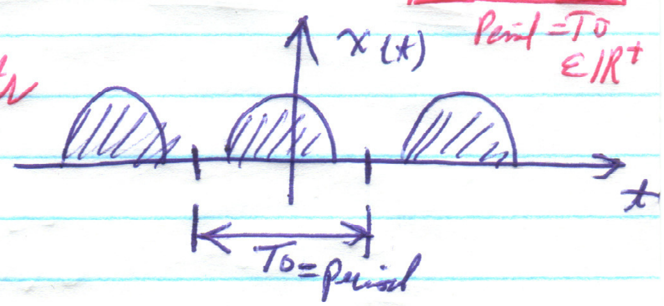
dummy Variable

F.S. coefficients

fundamental Frequency

$\boxed{\omega_0 = 2\pi/T_0}$

Period $= T_0 \in \mathbb{R}^+$



$k=0 \Longrightarrow$ $$\boxed{a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt}$$

$k \neq 0 \Longrightarrow$ $$\boxed{a_k = \frac{1}{T_0}\int_{T_0} x(t)\,e^{-jk\omega_0 t}\,dt}$$

Why do we even care about "a_k"?

Because "a_k" will enlighten us about the "frequency" behavior of a signal!

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⊙ F.S. for $x[n] \rightarrow$ Periodic:

$$\boxed{x[n] = \sum_{k=\langle N_0 \rangle} a_k \, e^{jk\omega_0 n}}$$

Fundamental frequency

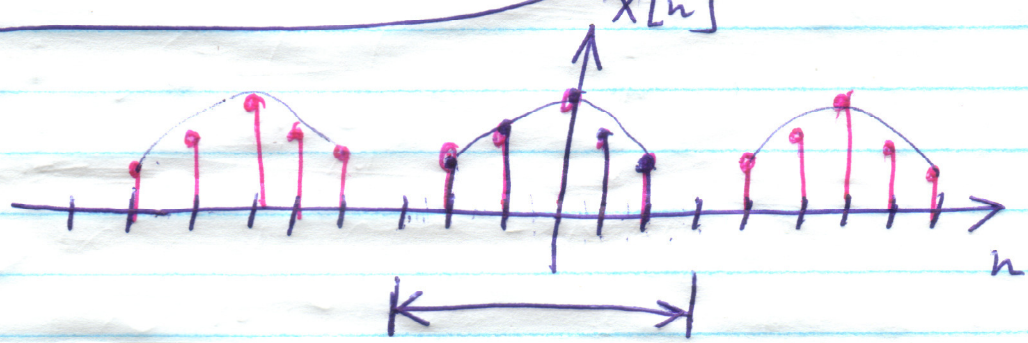
$$\boxed{\omega_0 = 2\pi / N_0}$$

Period $= N_0 =$ integer

dummy Variable

i.e. look only at one period of $x[n]$ (not the entire thing!!!)

F.S. coefficients



one period $= N_0 = 6$

$k = 0 \Longrightarrow$ $$\boxed{a_0 = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n]}$$

$k \neq 0 \Longrightarrow$ $$\boxed{a_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n] \, e^{-jk\omega_0 n}}$$

The end!

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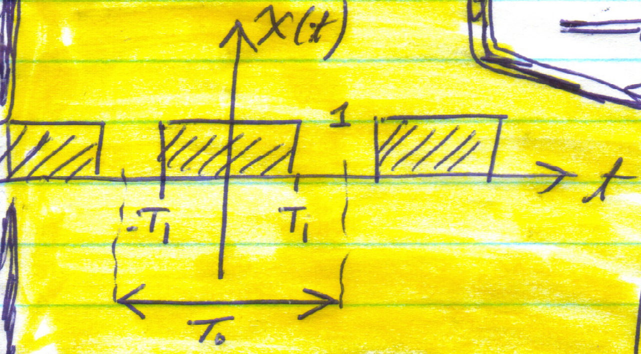
Some Useful Formulas And Properties



$\Longrightarrow$ F.S.

$$a_0 = (2T_1/T_0)$$

$$a_k = \frac{\sin(2\pi k T_1/T_0)}{\pi k}$$

$$\cos(\theta) = \frac{(e^{j\theta} + e^{-j\theta})}{2}$$

$$\sin(\theta) = \frac{(e^{j\theta} - e^{-j\theta})}{2j}$$

Similar property in Freq. Domain

$$e^{jM\omega_0 t} x(t) \longleftrightarrow a_{k-M}$$

i.e. shift the Frequency Domain to the right by "M" units

$\boxed{\text{integer}}$

Properties of Continuous-Time Fourier Series

$x_1(t) \xleftrightarrow{F.S.} a_k$

$x_2(t) \xleftrightarrow{F.S.} b_k$

$\omega_0 = \frac{2\pi}{T}$

Linearity	$Ax_1(t) + Bx_2(t)$	$Aa_k + Bb_k$
Time Shift	$x(t-t_0)$	$a_k e^{-jk\omega_0 t_0}$
Time Reversal	$x(-t)$	a_{-k}
Differentiation	$\frac{dx(t)}{dt}$	$a_k(jk\omega_0)$
Integration	$\int_{-\infty}^{t} x(t)\,dt$	$a_k\left(\frac{1}{jk\omega_0}\right)$
Periodic Convolution	$\int_T x_1(\tau)x_2(t-\tau)\,d\tau$	$T a_k b_k$

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Jan-02-2013

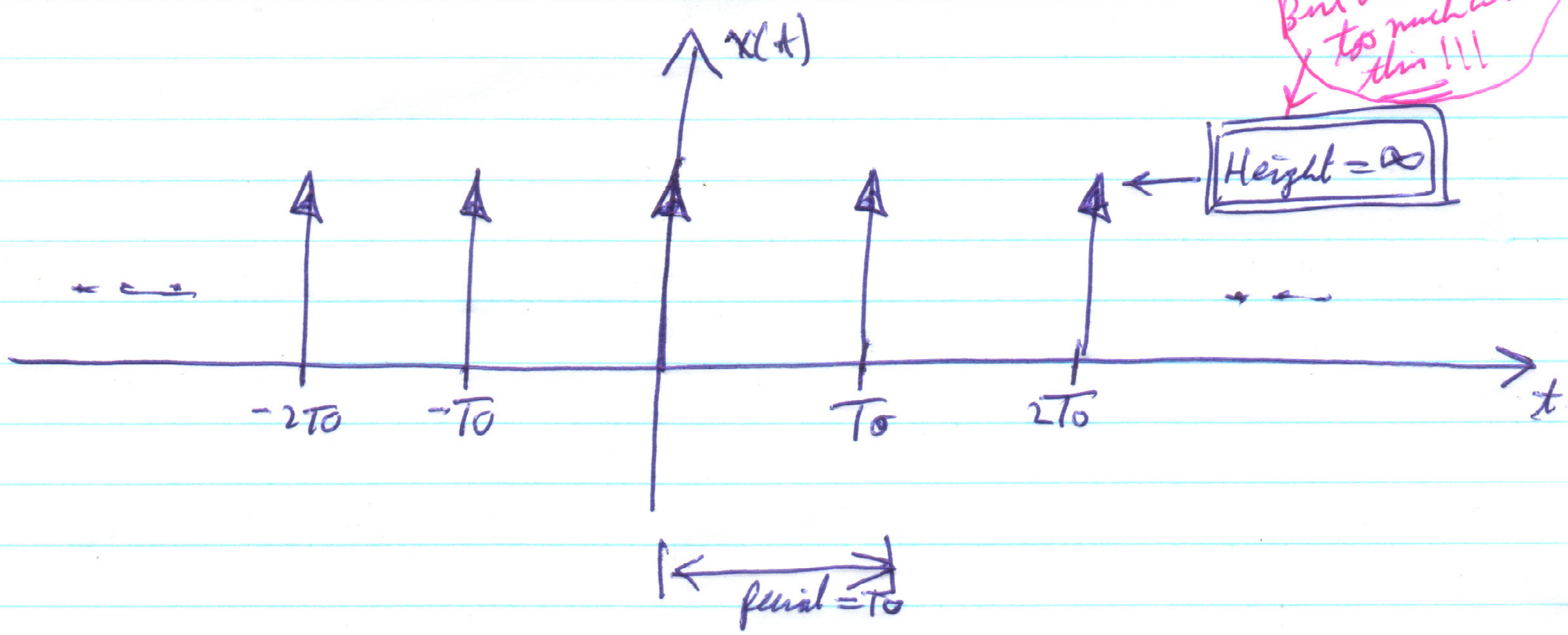
In addition to a "square wave signal" a signal of impulses or "impulse train" is also very common in DSP

∴ What is its "F.S." ?

$$\boxed{x(t) = \sum_{l=-\infty}^{\infty} \delta(t - l T_0)}$$ ← impulse Train Signal

period = T_0

But don't think too much about this!!!



$$\boxed{a_0} = \frac{1}{T_0} \int_{T_0} x(t)\, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t)\, dt = \boxed{\frac{1}{T_0}}$$

$$\underset{k \neq 0}{\boxed{a_k}} = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \underbrace{\delta(t) \quad e^{-jk\omega_0 t}}\, dt.$$ → Next Page

$$\boxed{\delta(t - t_0) \cdot Z(t) = \delta(t - t_0) \cdot Z(t_0)}$$ ← useful Fact!!

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Jun-02-2012

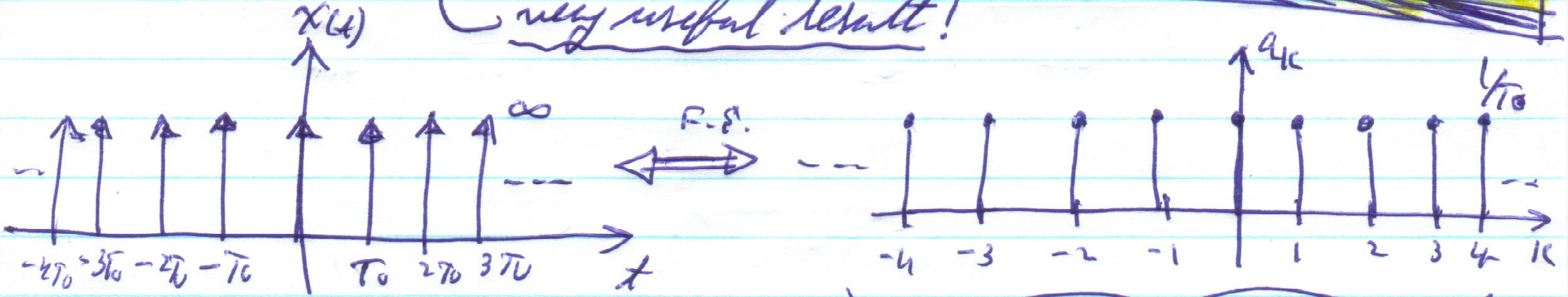
$$\therefore \underbrace{\delta(t)}_{\substack{\text{only valid}\\ \text{at } t=0}} \cdot e^{-jk\omega_0 t} = \delta(t)\cdot e^{-jk\omega_0 (0)}$$

$$= \delta(t)\cdot e^{0} = \boxed{\delta(t)}$$

$$\therefore \boxed{\begin{matrix} a_K \\ K \neq 0 \end{matrix}} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\,dt = \boxed{\frac{1}{T_0}}$$

$$\therefore \boxed{x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)} \xleftrightarrow{\text{F.S.}} \boxed{\begin{matrix} a_K = \dfrac{1}{T_0} \\ K = 0, \pm 1, \pm 2, \pm 3, \dots \end{matrix}}$$

very useful result!



continuous impulse train | discrete impulse train

$\boxed{\text{Time}}$

a train gives a train!!!

$\boxed{\text{Frequency}}$